Goldstein 1.9

In cgs units,

$$L = T - q\phi + \frac{1}{c}\vec{v}\cdot\vec{A} \quad \text{and} \quad \vec{F} = q\vec{E} + \frac{\vec{v}}{c}\times\vec{B}$$

Suppose

$$\vec{A} \to \vec{A} + \vec{\nabla}\psi(\vec{r},t)$$

$$\phi \to \phi - \frac{1}{c}\frac{\partial\psi}{\partial t} \qquad \text{for arbitrary } \psi.$$

The new Lagrangian is:

$$L' = T - q\left(\phi - \frac{1}{c}\frac{\partial\psi}{\partial t}\right) + \frac{q}{c}\left(\vec{A} + \vec{\nabla}\psi\right)\cdot\vec{v}$$

$$= T - q\phi + \frac{q}{c}\vec{A}\cdot\vec{v} + \frac{q}{c}\left(\frac{\partial\psi}{\partial t} + \vec{\nabla}\psi\cdot\vec{v}\right)$$

$$= L + \frac{q}{c}\frac{d\psi}{dt}$$

so, using the result of G 1.8, the equations of motion are unaffected. The new Hamiltonian is:

$$H' = H + \frac{q}{c}\vec{\nabla}\psi\cdot\vec{v} - \frac{q}{c}\frac{d\psi}{dt}$$

$$= H - \frac{q}{c}\frac{\partial\psi}{\partial t}$$

and the equations of motion are again the same, using 8.19.