old Goldstein

Goldstein 1.8 (and 8.19)

Suppose $L' = L + \frac{dF}{dt}$ where $F = F(q_1, q_2, \cdots q_n, t)$.

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_i}\left(\sum \frac{\partial F}{\partial q_j}\dot{q}_j + \frac{\partial F}{\partial t}\right)\right]$$

$$= \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \frac{d}{dt}\left[\sum \frac{\partial F}{\partial q_j}\delta_{ij}\right]$$

$$= \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \frac{d}{dt}\frac{\partial F}{\partial q_i}$$

$$= \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \sum \frac{\partial^2 F}{\partial q_k \partial q_i}\dot{q}_k + \frac{\partial^2 F}{\partial t \partial q_i}$$

$$= \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \frac{\partial}{\partial q_i}\left(\frac{dF}{dt}\right)$$

and

$$\frac{\partial L'}{\partial q_i} = \frac{\partial L}{\partial q_i} + \frac{\partial}{\partial q_i}\left(\frac{dF}{dt}\right)$$

so that

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}_i} - \frac{\partial L'}{\partial q_i} = \overbrace{\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}\right)}^{=0} + \cancel{\frac{\partial}{\partial q_i}\left(\frac{dF}{dt}\right)} - \cancel{\frac{\partial}{\partial q_i}\left(\frac{dF}{dt}\right)} = 0$$

so L' satisfies the Lagrange equations if L does.

Old Goldstein 8.19

19. It has been previously noted that the total time derivative of a function of q_i and t can be added to the Lagrangian without changing the equations of motion. What does such an addition do to the canonical momenta and the Hamiltonian? Show that the equations of motion in terms of the new Hamiltonian reduce to the original Hamilton's equations of motion.