Goldstein 1.10

$L = L(q, \dot{q}, t)$

$q_i = q_i(s_1, \dots, s_n, t) \qquad i = 1, \dots, n$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

$$\frac{\partial L}{\partial s_i} = \sum_{k=1}^{n} \frac{\partial L}{\partial q_k}\frac{\partial q_k}{\partial s_i} + \sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}_k}\frac{\partial \dot{q}_k}{\partial s_i}$$

$$\frac{\partial L}{\partial \dot{s}_i} = \sum_{k=1}^{n} \frac{\partial L}{\partial q_k}\cancel{\frac{\partial q_k}{\partial \dot{s}_i}}^{\,0} + \sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}_k}\frac{\partial \dot{q}_k}{\partial \dot{s}_i}$$

But $\dot{q}_k = \sum_{j=1}^{n} \frac{\partial q_k}{\partial s_j}\dot{s}_j + \frac{\partial q_k}{\partial t} \implies \frac{\partial \dot{q}_k}{\partial \dot{s}_j} = \frac{\partial q_k}{\partial s_j}$ so:

$$\frac{\partial L}{\partial \dot{s}_i} = \sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}_k}\frac{\partial q_k}{\partial s_i}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}_i}\right) = \sum_{k=1}^{n} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\frac{\partial q_k}{\partial s_i} + \sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}_k}\frac{d}{dt}\left(\frac{\partial q_k}{\partial s_i}\right)$$

we also have: $\frac{d}{dt}\left(\frac{\partial q_i}{\partial s_j}\right) = \sum_{k=1}^{n} \frac{\partial^2 q_i}{\partial s_k \partial s_j}\dot{s}_k + \frac{\partial^2 q_i}{\partial s_j \partial t} = \frac{\partial}{\partial s_j}\left(\sum_{k=1}^{n} \frac{\partial q_i}{\partial s_k}\dot{s}_k + \frac{\partial q_i}{\partial t}\right)$

$$\frac{d}{dt}\left(\frac{\partial q_i}{\partial s_j}\right) = \frac{\partial \dot{q}_i}{\partial s_j}$$

Therefore:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}_i}\right) - \frac{\partial L}{\partial s_i} = \sum_{k=1}^{n}\left[\frac{\partial q_k}{\partial s_i}\overbrace{\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k}\right)}^{=0} + \cancel{\frac{\partial L}{\partial \dot{q}_k}\frac{\partial \dot{q}_k}{\partial s_i}} - \cancel{\frac{\partial L}{\partial \dot{q}_k}\frac{\partial \dot{q}_k}{\partial s_i}}\right]$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}_i}\right) - \frac{\partial L}{\partial s_i} = 0$$