D C.M. 1.21

a) Equation (5.42) reads $U = -\frac{G}{2} \sum_{i \neq j} \frac{m_i m_j}{\| \vec{r}_i - \vec{r}_j \|}$.

In the continuum limit this expression becomes $U = -\frac{G}{2} \iint d^3\vec{r}\, d^3\vec{r}\,' \frac{\rho(\vec{r})\rho(\vec{r}\,')}{\| \vec{r} - \vec{r}\,' \|}$.

Rearranging the integrals $\Rightarrow$

$$U = \frac{1}{2} \int d^3\vec{r}\, \rho(\vec{r}) \int \frac{d^3\vec{r}\,'(-G)\rho(\vec{r}\,')}{\| \vec{r} - \vec{r}\,' \|}$$

$$\Rightarrow \boxed{U = \frac{1}{2} \int d^3\vec{r}\, \rho(\vec{r})\, \phi(\vec{r})} \quad \text{where} \quad \boxed{\phi(\vec{r}) = -G \int \frac{d^3\vec{r}\,' \rho(\vec{r}\,')}{\| \vec{r} - \vec{r}\,' \|}}$$

-vidently $\phi(\vec{r})$ satisfies the "Poisson" equation

$\nabla^2 \phi(\vec{r}) = 4\pi G \rho(\vec{r})$ and the "divergence" equation

$\nabla \vec{g} = -4\pi G \rho(\vec{r})$ where $\vec{g} = -\nabla\phi$.

Consider a sphere of uniform density ρ_0 with radius R, and construct a Gaussian shell of radius r $\Rightarrow$

$$\underset{\substack{\text{surface} \\ \text{of shell}}}{\int} \vec{g} \cdot d\vec{A} = \underset{\substack{\text{volume} \\ \text{of shell}}}{\int} \nabla \cdot \vec{g}\; d^3\vec{r}\,'$$