DLN
2.3.1

$$\vec{a} = h\vec{f}^{\,n}$$
$$\vec{b} = h\vec{f}(\vec{y}^{\,n}+\vec{a},\ t^n+h)$$
$$\vec{y}^{\,n+1} = \vec{y}^{\,n} + \tfrac{1}{2}(\vec{a}+\vec{b}).$$

(1) Taylor says
$$\vec{y}^{\,n+1}_{\text{true}} = \vec{y}^{\,n} + h\frac{d}{dt}\vec{y}^{\,n} + \tfrac{1}{2}h^2\frac{d^2}{dt^2}\vec{y}^{\,n} + \tfrac{1}{3!}h^3\frac{d^3}{dt^3}\vec{y}^{\,n} + O(h^4)$$

Expanding $\vec{b}$ gives
$$\vec{b} = h\Big[\vec{f}(\vec{y}^{\,n},t^n) + h\check{\vec{f}}(\vec{y}^{\,n},t^n) + \sum_k a_k\frac{\partial\vec{f}}{\partial y_k}(\vec{y}^{\,n},t^n) + \tfrac{1}{2}h^2\check{\check{\vec{f}}}(\vec{y}^{\,n},t^n) + \tfrac{1}{2}\sum_{k,l}a_k a_l\frac{\partial^2\vec{f}}{\partial y_k\partial y_l}(\vec{y}^{\,n},t^n) + \sum_k a_k h\frac{\partial\check{\vec{f}}}{\partial y_k}(\vec{y}^{\,n},t^n)\Big] + O(h^4).$$

Since $\vec{a} = O(h)$, this includes all $O(h^3)$ terms

** Also note $\check{f}^n = \partial f^n/\partial t$ <u>not</u> $df^n/dt$ **

$$\vec{b} = h\Big[\vec{f}^{\,n} + h\check{\vec{f}}^{\,n} + h f^n_k\vec{f}^{\,n}_{,k} + \tfrac{1}{2}h^2\check{\check{\vec{f}}}^{\,n} + \tfrac{1}{2}h^2 f^n_k f^n_l\vec{f}^{\,n}_{,kl} + h^2 f^n_k\check{\vec{f}}^{\,n}_{,k}\Big] + O(h^4)$$

I have written $\vec{f}^{\,n}_{,k}$ for $\partial\vec{f}^{\,n}/\partial y_k$ and summation upon repeated <u>subscripts</u> is implied.

Runge-Kutta gives $\vec{y}^{\,n+1}_{\text{approx}} = \vec{y}^{\,n} + \tfrac{1}{2}(\vec{a}+\vec{b}) \Rightarrow$

(2)
$$\vec{y}^{\,n+1}_{\text{approx}} = \vec{y}^{\,n} + \tfrac{1}{2}\Big[2h\vec{f}^{\,n} + h^2\dot{\vec{f}}^{\,n} + \tfrac{1}{2}h^3\dddot{\vec{f}}^{\,n} + h^2 f^n_k\vec{f}^{\,n}_{,k} + h^3 f^n_k\dot{\vec{f}}^{\,n}_{,k} + \tfrac{1}{2}h^3 f^n_k f^n_l\vec{f}^{\,n}_{,kl}\Big] + O(h^4)$$

Using $\dot{\vec{y}} = \vec{f}$

(1) $\rightarrow$
$$\vec{y}^{\,n+1}_{\text{true}} = \vec{y}^{\,n} + h\vec{f}^{\,n} + \tfrac{1}{2}h^2\frac{d\vec{f}^{\,n}}{dt} + \tfrac{1}{6}h^3\frac{d^2\vec{f}^{\,n}}{dt^2} + O(h^4)$$

Chain rule
$$\frac{d\vec{f}^{\,n}}{dt} = \check{\vec{f}}^{\,n} + \frac{dy_k}{dt}\vec{f}^{\,n}_{,k}$$