D.L.N. 2.2.1

(a) Write $\ddot{x} + x = 0$ as

$$\left.\begin{array}{l} y_1 = x \\ y_2 = \dot{x} \end{array}\right\} \Rightarrow \begin{array}{ll} \dot{y}_1 = y_2 & = f_1(\vec{y}, t) \\ \dot{y}_2 = -y_1 & = f_2(\vec{y}, t) \end{array}$$

Euler: $\vec{y}^{\,n+1} = \vec{y}^{\,n} + h\vec{f}^{\,n} \longrightarrow$

$$\begin{array}{l} y_1^{n+1} = y_1^n + h y_2^n \\ y_2^{n+1} = y_2^n - h y_1^n \end{array} \qquad \text{or}$$

(b) $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^{n+1} = \begin{pmatrix} 1 & h \\ -h & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^{n}$ which gives

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^{n} = M^n \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^{0} \qquad M = \begin{pmatrix} 1 & h \\ -h & 1 \end{pmatrix}$$

(c) To find eigenvalues & eigenvectors of $M$:

$$0 = \begin{vmatrix} 1-\lambda & h \\ -h & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + h^2 = \lambda^2 - 2\lambda + 1 + h^2$$

$$\lambda = 1 \pm ih$$

$$\alpha = 1 + ih \qquad \beta = 1 - ih$$

$a$: $\begin{pmatrix} -ih & h \\ -h & -ih \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \qquad \begin{array}{l} -ia_1 + a_2 = 0 \\ a_2 = ia_1 \end{array}$

$$\vec{a} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{similarly} \quad \vec{b} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

[you may complex conjugate both sides of $M\vec{a} = \alpha\vec{a}$, since $M$ is real].