Continued Fraction

6.1.48

$$\ln \Gamma(z)+z-(z-\tfrac{1}{2})\ln z-\tfrac{1}{2}\ln(2\pi)$$
$$=\frac{a_0}{z+}\,\frac{a_1}{z+}\,\frac{a_2}{z+}\,\frac{a_3}{z+}\,\frac{a_4}{z+}\,\frac{a_5}{z+}\,\dots \qquad (\mathscr{R}z>0)$$

$$a_0=\frac{1}{12},\ a_1=\frac{1}{30},\ a_2=\frac{53}{210},\ a_3=\frac{195}{371},$$
$$a_4=\frac{22999}{22737},\ a_5=\frac{29944523}{19733142},\ a_6=\frac{109535241009}{48264275462}$$

Wallis' Formula [4]

6.1.49

$$\frac{2}{\pi}\int_0^{\pi/2}\binom{\sin}{\cos}^{2n} x\,dx=\frac{1\cdot3\cdot5\ \dots\ (2n-1)}{2\cdot4\cdot6\ \dots\ (2n)}$$
$$=\frac{(2n)!}{2^{2n}(n!)^2}=\frac{1}{2^{2n}}\binom{2n}{n}=\frac{\Gamma(n+\frac{1}{2})}{\pi^{\frac{1}{2}}\Gamma(n+1)}$$
$$\sim\frac{1}{\pi^{\frac{1}{2}}n^{\frac{1}{2}}}\left[1-\frac{1}{8n}+\frac{1}{128n^2}-\dots\right]$$
$$(n\to\infty)$$

Some Definite Integrals

6.1.50

$$\ln\Gamma(z)=\int_0^\infty\left[(z-1)\,e^{-t}-\frac{e^{-t}-e^{-zt}}{1-e^{-t}}\right]\frac{dt}{t}\qquad(\mathscr{R}z>0)$$
$$=(z-\tfrac{1}{2})\ln z-z+\tfrac{1}{2}\ln 2\pi$$
$$+2\int_0^\infty\frac{\arctan(t/z)}{e^{2\pi t}-1}\,dt\qquad(\mathscr{R}z>0)$$

6.2. Beta Function

6.2.1

$$B(z,w)=\int_0^1 t^{z-1}(1-t)^{w-1}\,dt=\int_0^\infty\frac{t^{z-1}}{(1+t)^{z+w}}\,dt$$
$$=2\int_0^{\pi/2}(\sin t)^{2z-1}(\cos t)^{2w-1}\,dt$$
$$(\mathscr{R}z>0,\ \mathscr{R}w>0)$$

6.2.2 $$B(z,w)=\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}=B(w,z)$$

6.3. Psi (Digamma) Function [5]

6.3.1 $$\psi(z)=d[\ln\Gamma(z)]/dz=\Gamma'(z)/\Gamma(z)$$

[4] Some authors employ the special double factorial notation as follows:

$(2n)!!=2\cdot4\cdot6\ \dots\ (2n)=2^n n!$

$(2n-1)!!=1\cdot3\cdot5\ \dots\ (2n-1)=\pi^{-\frac{1}{2}}2^n\,\Gamma(n+\tfrac{1}{2})$

[5] Some authors write $\psi(z)=\frac{d}{dz}\ln\Gamma(z+1)$ and similarly for the polygamma functions.

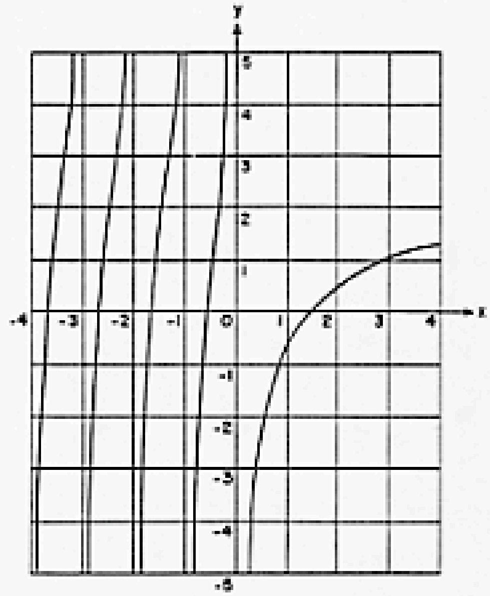

FIGURE 6.2. *Psi function.*

$y=\psi(x)=d\ln\Gamma(x)/dx$

Integer Values

6.3.2 $$\psi(1)=-\gamma,\ \psi(n)=-\gamma+\sum_{k=1}^{n-1}k^{-1}\qquad(n\geq2)$$

Fractional Values

6.3.3

$$\psi(\tfrac{1}{2})=-\gamma-2\ln 2=-1.96351\ 00260\ 21423\ \dots$$

6.3.4

$$\psi(n+\tfrac{1}{2})=-\gamma-2\ln 2+2\left(1+\frac{1}{3}+\dots+\frac{1}{2n-1}\right)$$
$$(n\geq1)$$

Recurrence Formulas

6.3.5 $$\psi(z+1)=\psi(z)+\frac{1}{z}$$

6.3.6

$$\psi(n+z)=\frac{1}{(n-1)+z}+\frac{1}{(n-2)+z}+\dots$$
$$+\frac{1}{2+z}+\frac{1}{1+z}+\psi(1+z)$$