Thus, since the $\hat{f}_b^*$ with $b = 0, \pm$ are all eigenvectors of the moment of inertia tensor with the same eigenvalue, so is any linear combination of the $\hat{f}_b^*$, and hence any axis through the CM in the $\hat{e}_1^*, \hat{e}_2^*$ plane is a principal axis. In particular, $I_{11} = I_{22}$. Let us compute I_{11}:

$$I_{11} = \int d^3\vec{r}\, \rho(\vec{r}) \left\{ \vec{r}^{\,2} - x^2 \right\} = m A^{-1} \int_{\text{triangle}} dx\,dy\; y^2 . \quad \text{Let } \xi = x - \frac{a}{2\sqrt{3}} .$$

$$\text{Then } I_{11} = m A^{-1}\, 2 \int_0^{\frac{\sqrt{3}}{2}a} d\xi \int_0^{\frac{a}{2}\left(1 - \frac{2}{\sqrt{3}}\frac{\xi}{a}\right)} y^2\,dy = \frac{2 m A^{-1}}{3} \left(\frac{a}{2}\right)^3 \int_0^{\frac{\sqrt{3}}{2}a} \left(1 - \frac{2}{\sqrt{3}}\xi/a\right)^3 d\xi$$

$$= \frac{2 m A^{-1}}{3} \left(\frac{a}{2}\right)^3 \int_0^1 (1-\eta)^3\, \frac{\sqrt{3}}{2} a\, d\eta = \frac{m A^{-1} a^4}{8\sqrt{3}} \int_0^1 d\eta \left\{ 1 - 3\eta + 3\eta^2 - \eta^3 \right\}$$

Let $\xi = \frac{\sqrt{3}}{2} a \eta$

$$= m \frac{a^4}{8\sqrt{3}} \left[\frac{a}{2} \frac{\sqrt{3}}{2} a \right]^{-1} \left[\eta - \frac{3\eta^2}{2} + \frac{3\eta^3}{3} - \frac{\eta^4}{4} \right] \Big|_0^1$$

$$= \frac{m a^2}{2 \otimes 3} \left[1 - \frac{3}{2} + 1 - \frac{1}{4} \right] = \boxed{\frac{m a^2}{24} = I_{11} = I_{22}}$$