Dragt 8.90 cont.

$$\dot{\hat{e}}_1^{*} = (-\hat{e}_1 \sin\phi - \hat{e}_2 \cos\phi)\,\dot{\phi} = \vec{\omega} \times \hat{e}_1^{*}$$

$$\dot{\hat{e}}_2^{*} = (\hat{e}_1 \cos\phi - \hat{e}_2 \sin\phi)\,\dot{\phi} = \vec{\omega} \times \hat{e}_2^{*}$$

$$\dot{\hat{e}}_3^{*} = 0 = \vec{\omega} \times \hat{e}_3^{*} \quad \Longrightarrow \quad \vec{\omega} = -\dot{\phi}\,\hat{e}_3 = -\dot{\phi}\,\hat{e}_3^{*} \quad \therefore \quad \boxed{\begin{matrix} \omega_1^{*} = 0 & \omega_3^{*} = -\dot{\phi} \\ \omega_2^{*} = 0 & \end{matrix}}$$

Now use

$$T = \frac{1}{2}\left(\frac{M}{2}\right)\left(\dot{\vec{R}}^{cM}\right)^2 + \frac{1}{2}\,\vec{\omega}\cdot\vec{\vec{I}}^{cM}\cdot\vec{\omega} = \frac{M}{4} R^2 \dot{\phi}^2\left[\left(1-\frac{3}{8}\cos\phi\right)^2 + \left(\frac{3}{8}\sin\phi\right)^2\right] + \frac{1}{2}\dot{\phi}^2 I_{33}^{cM}$$

$$V = \frac{M}{2} g R_2^{cM} = \frac{MgR}{2}\left(1 - \frac{3}{8}\cos\phi\right).$$ For small oscillations, $\phi \approx 0 \Rightarrow$

$$T \simeq \frac{M}{4} R^2 \left(\frac{5}{8}\right)^2 \dot{\phi}^2 + \frac{1}{2}\dot{\phi}^2 I_{33}^{cM} + O(\phi^2\dot{\phi}^2), \qquad V \simeq \frac{MgR}{2}\left(1 - \frac{3}{8} + \frac{3}{16}\phi^2\right) + O(\phi^4).$$

$\therefore$ For small oscillations, $$T \simeq \frac{\dot{\phi}^2 MR^2}{2}\left[\frac{25}{128} + \frac{1}{5} - \frac{9}{128}\right] = \frac{\dot{\phi}^2 MR^2}{2}\,\frac{13}{40}$$

$$\therefore \mathcal{L}^{eff} = T - V = \frac{1}{2} MR^2 \dot{\phi}^2 \frac{13}{40} - \frac{5}{8}\frac{MgR}{2} - \frac{MgR}{2}\frac{3}{16}\phi^2$$

$$\frac{\partial \mathcal{L}^{eff}}{\partial \dot{\phi}} = \frac{13}{40} MR^2 \dot{\phi}, \quad \frac{\partial \mathcal{L}}{\partial \phi} = -MgR\frac{3}{16}\phi \quad \Rightarrow \quad \frac{13}{40} MR^2 \ddot{\phi} + MgR\frac{3}{16}\phi = 0$$

or $$\ddot{\phi} + \frac{g}{R}\frac{3}{16}\frac{40}{13}\phi = 0 \Rightarrow \Omega^2 = \frac{g}{R}\frac{3\times 40}{13\times 16} = \frac{g}{R}\frac{15}{26} \Rightarrow \boxed{\Omega = \sqrt{\frac{g}{R}\frac{15}{26}}}$$