$$L = \sum_j p_j \dot{q}_j - \mathcal{H}, \quad \dot{q}_1 = \frac{\partial \mathcal{H}}{\partial p_1} = p_1 + t, \quad \dot{q}_2 = \frac{\partial \mathcal{H}}{\partial p_2} = 2p_2$$

$$L = p_1(p_1+t) + p_2(2p_2) - \mathcal{H} = p_1^2 + p_1 t + 2p_2^2 - \mathcal{H}$$
$$= \tfrac{1}{2} p_1^2 + p_2^2 - q_1^3 - t q_2^2 = \tfrac{1}{2}(\dot{q}_1 - t)^2 + (\dot{q}_2/2)^2 - q_1^3 - t q_2^2$$

PART I (40 points)

1. (4 pts) A system with two degrees of freedom has the peculiar Hamiltonian $H(q,p,t) = \frac{1}{2}p_1^2 + tp_1 + p_2^2 + q_1^3 + tq_2^2$. Its Lagrangian is given explicitly by

$$L(q,\dot{q},t) = \tfrac{1}{2}(\dot{q}_1 - t)^2 + (1/4)\dot{q}_2^2 - q_1^3 - t q_2^2$$

or

$$L = \tfrac{1}{2}\dot{q}_1^2 - \dot{q}_1 t + (1/4)\dot{q}_2^2 - q_1^3 - t q_2^2 + \tfrac{1}{2}t^2$$

2. (8 pts) Consider the functional

$$I[y] = \int_0^1 [\dot{y}^2 + y - t^3]dt.$$

Find $y(t)$, subject to the end point conditions $y(0) = 0$ and $y(1) = 1$, such that I is an extremum.

$$y(t) = (3/4)t + t^2/4$$

$$\delta I = 0 \Rightarrow \frac{d}{dt}\underbrace{\frac{\partial \mathcal{L}}{\partial \dot{y}}}_{= 2\dot{y}} - \underbrace{\frac{\partial \mathcal{L}}{\partial y}}_{= 1} = 0 \quad \text{with}$$

$$\mathcal{L} = \dot{y}^2 + y - t^3 \Rightarrow \frac{d}{dt}(2\dot{y}) - 1 = 0 \Rightarrow \ddot{y} = 1/2$$

$$\Rightarrow y = a + bt + (1/4)t^2$$

$y(0) = 0 \Rightarrow a = 0$

$y(1) = 1 \Rightarrow b + 1/4 = 1 \Rightarrow b = 3/4$