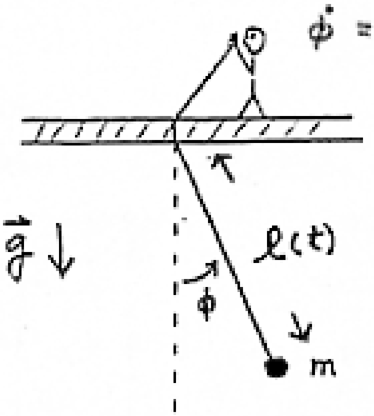
$$p_\phi = \frac{\partial L}{\partial \dot\phi} = m\ell^2\dot\phi \,, \quad \frac{\partial L}{\partial \phi} = -mg\ell\sin\phi \,, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot\phi} = m\ell^2\ddot\phi + 2m\ell\dot\ell\dot\phi$$

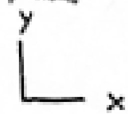
$$\Rightarrow \quad m\ell^2\ddot\phi + 2m\ell\dot\ell\dot\phi + mg\ell\sin\phi = 0 \,. \quad \mathcal{H} = p_\phi\dot\phi - \mathcal{L} \Rightarrow$$

$$\mathcal{H} = m\ell^2\dot\phi^2 - \tfrac{1}{2}m(\dot\ell^2 + \ell^2\dot\phi^2) - mg\ell\cos\phi \Rightarrow \mathcal{H} = \tfrac{1}{2}m\ell^2\dot\phi^2 - mg\ell\cos\phi - \tfrac{1}{2}m\dot\ell^2$$

10. (15 pts) Consider a simple plane pendulum whose length $\ell(t)$ is controlled by Ehrenfest's ghost.

$$\dot\phi = p_\phi / m\ell^2$$





$$x = \ell\sin\phi \,, \quad y = -\ell\cos\phi$$

$$\dot x = \dot\ell\sin\phi + \ell\cos\phi\,\dot\phi$$

$$\dot y = -\dot\ell\cos\phi + \ell\sin\phi\,\dot\phi$$

$$T = \tfrac{1}{2}m(\dot x^2 + \dot y^2) = \tfrac{1}{2}m(\dot\ell^2 + \ell^2\dot\phi^2)$$

Assume that $\ell(t)$ is a known function. $V = mgy = -mg\ell\cos\phi$

(a) Using ϕ as a generalized coordinate, write the Lagrangian for the pendulum.

$$L(\phi, \dot\phi, t) = \tfrac{1}{2}m(\dot\ell^2 + \ell^2\dot\phi^2) + mg\ell\cos\phi$$

(b) Find the differential equation of motion for $\phi(t)$.

$$m\ell^2\ddot\phi + 2m\ell\dot\ell\dot\phi + mg\ell\sin\phi = 0$$

or

$$\ddot\phi + 2(\dot\ell/\ell)\dot\phi + \frac{g}{\ell}\sin\phi = 0$$

(c) Find the Hamiltonian.

$$H(\phi, p_\phi, t) = \frac{p_\phi^2}{2m\ell^2} - mg\ell\cos\phi - \tfrac{1}{2}m\dot\ell^2$$