10. (20 pts) The logistic map $\mathcal{M}$ sends x to $\bar{x}$ by the rule

$$\bar{x} = \lambda x(1 - x).$$

Describe in a brief essay below what you have learned about the properties of the logistic map.

The map has fixed points x_e given by $x_e = 0$ and $x_e = (\lambda - 1)/\lambda$. For $\lambda \in [1, 4]$ $x_e = 0$ is unstable. For $\lambda \in [1, 3]$ $x_e = (\lambda - 1)/\lambda$ is stable, and it is unstable when $\lambda > 3$. Under repeated iteration of the map, period doubling occurs at $\lambda = \lambda_1 = 3$, and there is a cascade of successive period doublings at values λ_2, λ_3, $\cdots$. The cascade ends at a finite value λ_∞, and (apart from windows of stability) chaotic behavior occurs for $\lambda > \lambda_\infty$. The way in which the sequence λ_j converges to λ_∞ is universal. That is, a large class of maps behave in the same way. The way in which successive bifurcation forks scale is also universal. See the picture below. For $\lambda = 4$, the logistic map can be analyzed in detail, and it is found to contain the Bernoulli shift map. Thus, even a very simple map can lead to very complicated behavior under repeated iteration.

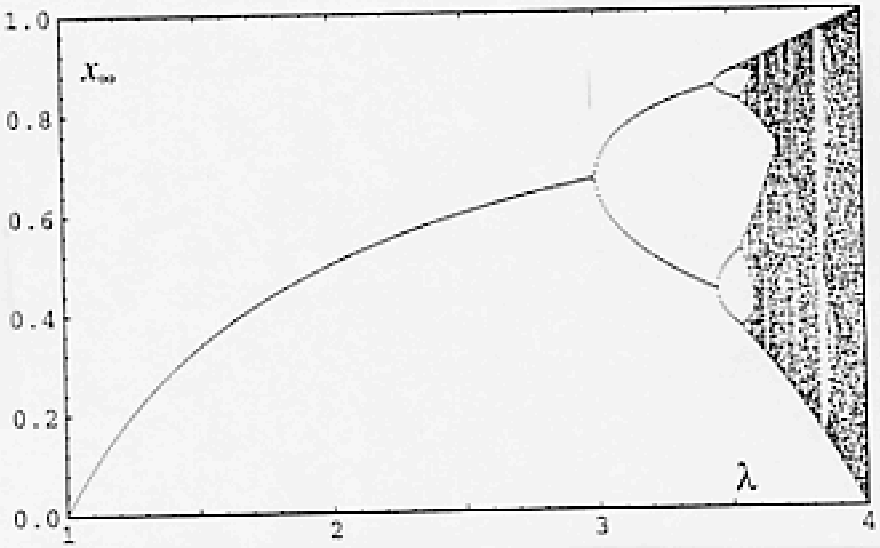

Figure 1.2.2: Feigenbaum diagram showing limiting values x_∞ as a function of λ for the logistic map.