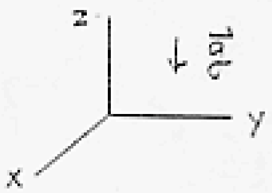
4. (4 pts) Consider a particle of mass m constrained to move without friction on the curve $x = 0$, $z = y^2$ under the influence of gravity.



$$\mathcal{L} = T - V, \quad T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$
$$\dot{x} = 0, \quad \dot{z} = 2y\dot{y}, \quad V = mgz$$
$$= mgy^2$$

Its Lagrangian is given by

$$\mathcal{L}(y, \dot{y}, t) = \frac{m}{2}(1 + 4y^2)(\dot{y})^2 - mgy^2$$

5. (3 pts) Consider a particle of mass m and charge q moving in an electromagnetic field described by the scalar and vector potential $\phi(\mathbf{r}, t)$ and $A(\mathbf{r}, t)$. Its Lagrangian in MKS units is given by

$$\mathcal{L}(\mathbf{r}, \mathbf{v}, t) = \frac{1}{2} m \vec{v} \cdot \vec{v} + q \vec{v} \cdot \vec{A}(\vec{r}, t) - q \phi(\vec{r}, t)$$

6. (6 pts) A system with 3 degrees of freedom has the peculiar Lagrangian

$$\begin{aligned}\mathcal{L}(q, \dot{q}, t) &= \dot{q}_1^2 q_2 \cos q_3 + \dot{q}_2^2 q_2 q_3 + \dot{q}_3^2 \tan(q_2 q_3) \\ &+ \dot{q}_1 \dot{q}_2 \dot{q}_3 + q_2 \exp q_3 + ct^5 q_2 q_3.\end{aligned}$$

$$\frac{\partial \mathcal{L}}{\partial q_1} = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{q}_1} = \text{cnst}$$

It has the constant of motion

$$p_1 = \frac{\partial \mathcal{L}}{\partial \dot{q}_1} = 2\dot{q}_1 q_2 \cos q_3 + \dot{q}_2 \dot{q}_3 = \text{cnst}$$

$$\frac{\partial \mathcal{L}}{\partial t} = 0 \Rightarrow \mathcal{H} = \text{cnst}$$

If $c = 0$, it has the additional constant of motion $\mathcal{H} = \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \dot{q}_i - \mathcal{L} \Rightarrow$

$$\mathcal{H} = (\dot{q}_1)^2 q_2 \cos q_3 + (\dot{q}_2)^2 q_2 q_3 + (\dot{q}_3)^2 \tan(q_2 q_3)$$

$$+ 2\dot{q}_1 \dot{q}_2 \dot{q}_3 - q_2 \exp q_3 = \text{cnst}$$