HW#1 Solutions

1) $\mathcal{L}(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega_0^2 x^2$

(a) $\frac{\partial \mathcal{L}}{\partial x} = -m\omega_0^2 x \quad , \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{d}{dt}(m\dot{x}) = m\ddot{x}$

$\Rightarrow$ EOM : $\boxed{\ddot{x} + \omega_0^2 x = 0}$

(b) $x = q^{1/3} \quad , \dot{x} = \frac{1}{3}q^{-2/3}\dot{q} \quad , \ddot{x} = -\frac{2}{9}q^{-5/3}\dot{q}^2 + \frac{1}{3}q^{-2/3}\ddot{q}$

$\frac{1}{3}q^{-2/3}\ddot{q} - \frac{2}{9}q^{-5/3}\dot{q}^2 + \omega_0^2 q^{1/3} = 0 \qquad \Big| \times 3q^{2/3}$

$\boxed{\ddot{q} - \frac{2}{3}\frac{\dot{q}^2}{q} + 3\omega_0^2 q = 0}$

(c) $\mathcal{L}(q,\dot{q}) = \frac{1}{2}m\left(\frac{1}{3}q^{-2/3}\dot{q}\right)^2 - \frac{1}{2}m\omega_0^2\left(q^{1/3}\right)^2$

$\boxed{\mathcal{L}(q,\dot{q}) = \frac{1}{18}m q^{-4/3}\dot{q}^2 - \frac{1}{2}m\omega_0^2 q^{2/3}}$

(d) $\frac{\partial \mathcal{L}}{\partial q} = -\frac{2}{27}m q^{-7/3}\dot{q}^2 - \frac{1}{3}m\omega_0^2 q^{-1/3} \quad , \quad \frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{1}{9}m q^{-4/3}\dot{q}$

$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \frac{1}{9}m q^{-4/3}\ddot{q} - \frac{4}{27}m q^{-7/3}\dot{q}^2$

$\Rightarrow$ EOM: $\frac{1}{9}m q^{-4/3}\ddot{q} - \frac{4}{27}m q^{-7/3}\dot{q}^2 + \frac{2}{27}m q^{-7/3}\dot{q}^2 + \frac{1}{3}m\omega_0^2 q^{-1/3} = 0 \qquad \Big| \times \frac{9}{m}q^{4/3}$

$\ddot{q} - \frac{4}{3}\frac{\dot{q}^2}{q} + \frac{2}{3}\frac{\dot{q}^2}{q} + 3\omega_0^2 q = 0$

$\boxed{\ddot{q} - \frac{2}{3}\frac{\dot{q}^2}{q} + 3\omega_0^2 q = 0}$

(e) They are the same. This is expected for the following reason: (This is the one I could come up with. There may be others!) The EOM is obtained by extremizing the action, which is a functional. Hence the problem has a unique answer irrespective of the choice of variables we use.

2) $\mathcal{L}(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega_0^2 x^2$; $x(0)=0,\ x(T)=\ell$

(a) $x(0,\omega) = \frac{\ell}{\sin(\omega T)}\sin(0) = 0$ ✓

$x(T,\omega) = \ell\,\frac{\sin(\omega T)}{\sin(\omega T)} = \ell$

(b) $\dot{x} = A\omega\cos\omega t$ where $A = \frac{\ell}{\sin(\omega T)}$

$\ddot{x} = -A\omega^2\sin\omega t$

EOM $\ddot{x} + \omega_0^2 x = -A\omega^2\sin\omega t + \omega_0^2 A\sin\omega t = 0$ satisfied if $\omega = \omega_0$ ✓

(c) $S = \int_0^T \mathcal{L}(x,\dot{x};t)\,dt = \int_0^T \left(\underbrace{\tfrac{1}{2}m\dot{x}^2}_{\text{integrate by parts}} - \tfrac{1}{2}m\omega_0^2 x^2\right)dt$

$= \frac{1}{2}m x\dot{x}\Big|_0^T - \underbrace{\int_0^T \left(\tfrac{1}{2}m x\ddot{x} + \tfrac{1}{2}m\omega_0^2 x^2\right)dt}_{\int_0^T \frac{1}{2}mx\,\underbrace{(\ddot{x}+\omega_0^2 x)}_{\text{This is zero! (using EOM) IF } \omega=\omega_0}\,dt}$

Valid for $\omega = \omega_0$ Only!

$= \frac{1}{2}m\,(A\sin\omega_0 t)(A\omega_0\cos\omega_0 t)$

$= \frac{1}{2}m\,\frac{\ell^2\omega_0}{\sin^2(\omega_0 T)}\sin\omega_0 t\cos\omega_0 t\Big|_0^T = \frac{1}{2}m\ell^2\omega_0\cot(\omega_0 T)$

If $\omega \neq \omega_0$ we have to perform the integral

$\int_0^T \frac{1}{2}m\,A\sin\omega t\left(-A\omega^2\sin\omega t + \omega_0^2 A\sin\omega t\right)dt = \frac{1}{2}mA^2\Big[\underbrace{\int_0^T \sin^2\omega t\,dt}_{I}\Big](\omega_0^2-\omega^2)$

$I = \int_0^T \frac{1}{2}(1-\cos 2\omega t)\,dt = \left(\frac{t}{2} - \frac{\sin(2\omega t)}{4\omega}\right)\Big|_0^T = \frac{T}{2} - \frac{\sin(2\omega T)}{4\omega}$

$$\Rightarrow \boxed{S(\omega) = \frac{1}{2}m\ell^2\omega\cot(\omega T) - \frac{1}{2}m\frac{\ell^2}{\sin^2(\omega T)}\left(\frac{T}{2} - \frac{\sin(2\omega T)}{4\omega}\right)(\omega_0^2-\omega^2)}$$

or $S(\omega) = \frac{m\ell^2}{8\omega\sin^2(\omega T)}\left[(\omega^2+\omega_0^2)\sin(2\omega T) + (\omega^2-\omega_0^2)2\omega T\right]$,

(d) $\left.\dfrac{dS(\omega)}{d\omega}\right|_{\omega_0} = 0$ can be shown by direct differentiation of the result obtained in part (c).

This is expected since the solution with $\omega = \omega_0$ is constructed in a way to extremize the action.

3) $$x(t,c) = \frac{\ell}{T}t + c\ell\left(\frac{t^3}{T^3} - \frac{t}{T}\right) = \overbrace{\left(\frac{\ell}{T} - \frac{c\ell}{T}\right)}^{a} t + \frac{c\ell}{T^3}t^3 \quad ; \; a = \frac{\ell}{T}(1-c)$$

$$\dot{x}(t,c) = a + \frac{3c\ell}{T^3}t^2$$

(b) $$S = \int_0^T dt \left\{ \frac{1}{2}m\left(a^2 + \frac{6c\ell}{T^3}at^2 + \frac{9c^2\ell^2}{T^6}t^4\right) - \frac{1}{2}m\omega_0^2\left[a^2t^2 + \frac{2c\ell}{T^3}at^4 + \frac{c^2\ell^2}{T^6}t^6\right]\right.$$

$$= \frac{1}{2}m\left\{ a^2T + 2c\ell a + \frac{9c^2\ell^2}{5T} - \omega_0^2\left[\frac{a^2}{3}T^3 + \frac{2c\ell a}{5}T^2 + \frac{c^2\ell^2}{7}T\right]\right\}$$

$$= \frac{1}{2}m\frac{\ell^2}{T}\left\{(1-c)^2 + 2c(1-c) + \frac{9c^2}{5}\right\} - \frac{1}{2}m\omega_0^2\ell^2 T\left\{\frac{(1-c)^2}{3} + \frac{2}{5}c(1-c) + \frac{c^2}{7}\right\}$$

$$\boxed{S = \frac{1}{2}m\frac{\ell^2}{T}\left\{1 + \frac{4}{5}c^2 - (\omega_0 T)^2\left(\frac{1}{3} - \frac{4}{15}c + \frac{8}{105}c^2\right)\right\}}$$

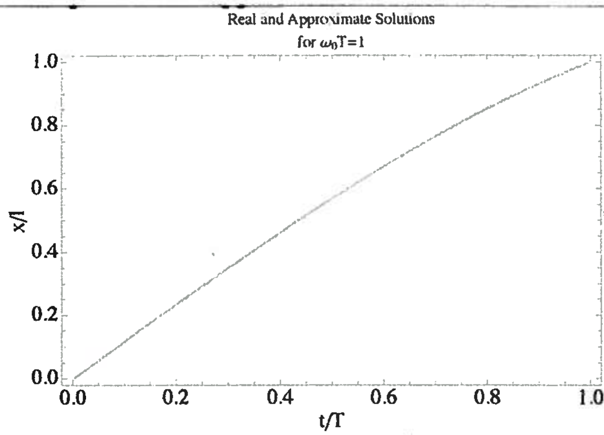
(c) $$\frac{dS}{dc} = \frac{1}{2}m\frac{\ell^2}{T}\left\{\frac{8}{5}c + (\omega_0 T)^2\left(\frac{4}{15} - \frac{16}{105}c\right)\right\} = 0$$

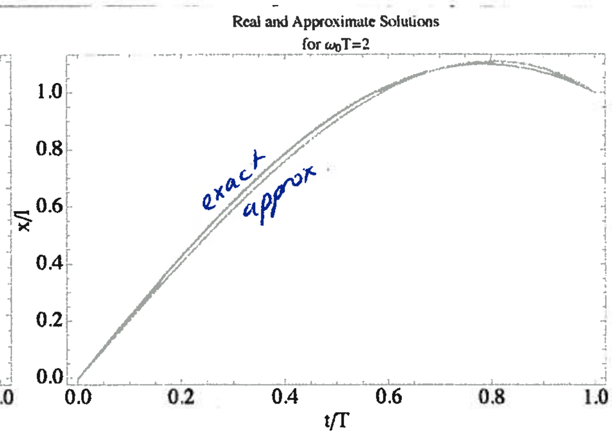
$$\Rightarrow \quad c\left(\frac{16}{105}(\omega_0 T)^2 - \frac{8}{5}\right) = (\omega_0 T)^2\frac{4}{15}$$

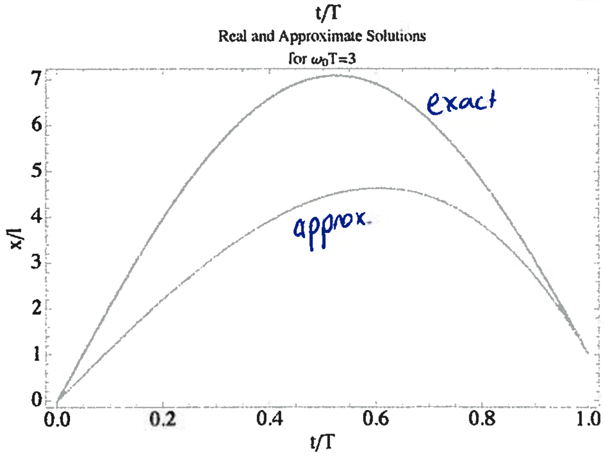
$$c = \frac{(\omega_0 T)^2\, 4/15}{\frac{16}{105}(\omega_0 T)^2 - \frac{8}{5}} = \boxed{\frac{\cancel{28}^{\,7}(\omega_0 T)^2}{\cancel{16}^{\,4}(\omega_0 T)^2 - \cancel{168}_{\,42}} = c}$$

(d) <u>Next page</u>

Next page







Note that the exact solution is a periodic function and the trial function is not. Hence we expect the trial function to perform worse at times longer than the period of motion $\sim 1/\omega_0$. Hence $\omega_0 T$ large means bad approximation as can be seen from the plots

Prob. Set #1

Jose & Salatan 2.11

A wire is bent into the shape given by $y = A|x^n|, n \geq 2$ and oriented vertically, opening upward, in a uniform gravitational field g. The wire rotates at a constant angular velocity ω about the y axis, and a bead of mass m is free to slide on it without friction.

(a) Find the equilibrium height of the bead on the wire. Consider especially the case n=2.

Equilibrium height occurs when the bead is at rest and its acceleration (second derivative wrt position) is zero. We know that in a non-rotating situation there is a stable equilibrium point at the bottom of the wire, so we can measure potential energy from there.

Constraint $f_1 = y - A|x^n| = 0$

The potential $V(y) = mgy = mgA|x^n|$

The kinetic energy $T(x, y) = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + x^2\omega^2)$ where $v_{\perp(aroundz)} = r\frac{d\phi}{dt} = x\omega$.

But $\dot{y} = An|x^{n-1}|\dot{x} \rightarrow \dot{y}^2 = A^2n^2|x^{2(n-1)}|\dot{x}^2$

Because we can write the potential in terms of x, we can impose the constraint directly with our choice of variables and write the Lagrangian only in terms of x and $\dot{x}$.

$L(x, \dot{x}) = T - V = \frac{1}{2}m(\dot{x}^2 + A^2n^2|x^{2(n-1)}|\dot{x}^2 + x^2\omega^2) - mgA|x^n|$

$\frac{\partial L}{\partial x} = m\omega^2x + mA^2n^2(n-1)|x^{2(n-1)-1}|\dot{x}^2 - mgAn|x^{n-1}|$

$\frac{\partial L}{\partial \dot{x}} = m\dot{x} + mA^2n^2|x^{2(n-1)}|\dot{x}$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x}(1 + A^2n^2|x^{2(n-1)}|) + mA^2n^2 2(n-1)|x^{2(n-1)-1}|\dot{x}^2$

$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$

$m\omega^2x + mA^2n^2(n-1)|x^{2(n-1)-1}|\dot{x}^2 - mgAn|x^{n-1}| - m\ddot{x}(1 + A^2n^2|x^{2(n-1)}|) - mA^2n^2 2(n-1)|x^{2(n-1)-1}|\dot{x}^2 = 0$

Simplifying,

$\ddot{x} = \frac{\omega^2x - gAn|x^{n-1}| - A^2n^2(n-1)|x^{2(n-1)-1}|\dot{x}^2}{(1 + A^2n^2|x^{2(n-1)}|)}$ This is a sloppy EOM.

To find the equilibrium points, let $\dot{x} = 0$ and set the acceleration to zero

$\ddot{x} = \frac{\omega^2x - gAn|x^{n-1}|}{(1 + A^2n^2|x^{2(n-1)}|)} = 0$

The equilibria occur when the numerator vanishes, that is when $\omega^2x = gAn|x^{n-1}|$.

Generally speaking, the equilibrium height is given by $\frac{x}{|x^{n-1}|} = \frac{gAn}{\omega^2}$ for $n > 0$.

In the case of n=2, the equilibrium height is at $y = x = 0$ -or- at ANY height when $\omega^2 = 2gA$.

(b) Find the frequency of small vibrations about the equilibrium position.

(b) Let $x(t) = x_{eq} + \eta(t)$ and substitute in the EOM.

Keep up to linear terms in η

$$\ddot{\eta} = \frac{\overbrace{\omega^2 x_{eq} + \omega^2 \eta - gAn\, x_{eq}^{n-1}}^{\text{cancellation by def. of } x_{eq}} - gAn(n-1)\, x_{eq}^{n-2}\, \eta}{1 + A^2 n^2 x_{eq}^{2(n-1)}} + O(\eta^2)$$

$$\ddot{\eta} + \frac{gAn(n-1)\, x_{eq}^{n-2} - \omega^2}{1 + A^2 n^2 x_{eq}^{2(n-1)}}\, \eta = 0 \qquad \text{use } x_{eq}^{n-2} = \frac{\omega^2}{gAn}$$

$$\ddot{\eta} + \underbrace{\frac{n-2}{1 + A^2 n^2 x_{eq}^{2(n-1)}}\, \omega^2}_{\Omega^2 \text{ : frequency of small oscillations.}} \eta = 0$$

For $n \leq 2$ the oscillations are unstable.

* Also for $n = 2$, $\Omega = 0$ which is expected since each point on the wire is an equilibrium point.

JS 3.2) (a) Without loss of generality let me put the starting position at the north pole and the final position at $\theta = \theta_f$, $\phi = \phi_f$.

Lagrangian in this case is only the kinetic energy term, which in spherical coordinates is given by (setting $\dot{R}=0$)

$$\mathcal{L} = \frac{mR^2}{2}\left(\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2\right)$$

Next derive the EOM's.

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial \mathcal{L}}{\partial \theta} \Rightarrow \ddot{\theta} = 2\sin\theta\cos\theta\, \dot{\phi}^2$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\partial \mathcal{L}}{\partial \phi} \Rightarrow \sin^2\theta\, \ddot{\phi} = 0$$

Note that these equations admit the following solution which satisfies the boundary conditions:

$$\phi = \text{constant} = \phi_f$$

$$\ddot{\theta} = 0 \Rightarrow \dot{\theta} = \text{constant} = \omega \Rightarrow \theta = \omega t$$

ω is determined by the requirement that at $t=T$ $\theta = \theta_f$

There are two solutions (Assuming particle leaves north pole at $t=0$ and arrives at θ_f, ϕ_f at time $t=T$ for the first time)

1) $\omega T = \theta_f \longrightarrow \omega_1 = \dfrac{\theta_f}{T}$

2) $\omega T = 2\pi - \theta_f \longrightarrow \omega_2 = \dfrac{2\pi - \theta_f}{T}$

* For the special cases of the final point being the north or the south pole there are uncountably many solutions (with ϕ value undetermined)

(b) $S = \int_0^{\pi} dt \left[\frac{mR^2}{2} (\omega t)^2 \right] = \frac{m\omega^2 R^2}{2} \frac{\pi^3}{3}$

The two ~~states~~ solutions have different actions as long as $\omega_1 \neq \omega_2$ or $\theta_f \neq \pi$ ~~(or 2π)~~ (and 2π depending on definition)

Nonphysical Paths

First note that the second term in the Lagrangian. $(\sim \sin^2\theta \dot{\phi}^2)$ which was set to zero for the physical solution, is positive definite. Hence this term will always increase the action it is enough for us to focus on the first term.

$$\theta(t) = \omega t + f(t) \qquad \text{where } f(0) = f(\pi) = 0$$

$$\dot{\theta}(t) = \omega + f'(t) \quad ; \quad \dot{\theta}^2(t) = \omega^2 + 2\omega f'(t) + [f'(t)]^2$$

$$\int_0^{\pi} \dot{\theta}^2 dt = \int_0^{\pi} \left(\omega^2 + [f']^2 + 2\omega f'\right) dt = S_{physical} + \underbrace{\int f'^2 dt}_{\text{positive}} + \underbrace{2\omega f \Big|_0^{\pi}}_{0}$$

$\Rightarrow$ The action is bigger for any nonphysical path.