PHYS420 (Spring 2006) Homework # 1 Solutions

Problem 1.

This problem deals with the expansion of a function, $f(x) = (1+x)^n$, into an infinite power series (i.e. $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$) where the coefficients of the successive terms of the series $(a_0, a_1, a_2, a_3, \ldots)$ involve the successive derivatives of the function. This type of expansion is known as Taylor's expansion and can be written as:

$$f(x) = \sum_{i=0}^{\infty} \frac{(x-a)^i}{i!} f^{(i)}(a) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots \quad (1)$$

In our particular example we want to expand our function for very small x; in other words, around x=0. So computing the first couple of terms, we get:

$$(1+x)^n = (1+0)^n + (x-0)n(1+0)^{n-1} + \frac{(x-0)^2}{2!} n(n-1)(1+0)^{n-2} + \ldots \quad (2)$$

$$= 1 + nx + \frac{1}{2} n(n-1)x^2 + \ldots \quad (3)$$

Keeping just the first term:

$$(1+x)^n \cong 1 + nx \quad (4)$$

Problem 2.

In order to show that Newton's 2nd law (F=ma) is valid in a moving frame, we must look at how accelerations transform.

(a) For a frame moving at constant speed v (in the positive x direction) relative to a stationary frame, the Galilean coordinate transformations is just

$$x' = x - vt \quad (5)$$

Taking a time derivative (d/dt') and recalling that dt'=dt and that v is a constant, we find how the velocity transforms. This is simply the Galilean velocity addition law.

$$u' = \frac{dx'}{dt'} = \frac{d}{dt'}(x - vt) = \frac{dx}{dt'} - v\frac{dt}{dt'} = \frac{dx}{dt} - v\frac{dt}{dt} = u - v \quad (6)$$

To find the acceleration we take another time derivative (d/dt', with $dt' = dt$).

$$a' = \frac{du'}{dt'} = \frac{d}{dt'}(u - v) = \frac{du}{dt'} = \frac{du}{dt} = a \quad (7)$$

And so we find that the accelerations are identical. Therefore so are the forces,

$$F' = ma' = ma = F \tag{8}$$

(b) For a frame moving at constant acceleration a_0 (in the positive x direction) relative to a stationary frame we cannot use the standard Galilean transformation rules that we derived in class anymore. Assuming that at time $t = t' = 0$ the two frames are together, then at some arbitrary time t later, the distance between the two frames will be $1/2a_0t^2$ in the x direction. In this case, the new coordinate transformation is

$$x' = x - \frac{1}{2}a_0t^2 \tag{9}$$

Taking time derivatives to find the velocity (and remembering that dt' = dt),

$$u' = \frac{dx'}{dt'} = \frac{d}{dt'}(x - \frac{1}{2}a_0t^2) = \frac{dx}{dt'} - a_0t\frac{dt}{dt'} = u - a_ot \tag{10}$$

Taking time derivatives to find the acceleration,

$$a' = \frac{du'}{dt'} = \frac{d}{dt'}(u - a_0t) = \frac{(}{d}u)(dt') - a_0\frac{dt}{dt'} = a - a_0 \tag{11}$$

We see that the two accelerations are not identical. So in this case, Newton's 2nd law will not have the same value in the two different frames

$$F' = ma' = m(a - a_0) = ma - ma_0 = F - ma_0! \tag{12}$$

Problem 3.

(i) In the rest frame:

In an elastic collision energy and momentum are conserved.

$$p_i = m_1v_{1i} + m_2v_{2i} = (0.3kg)(5m/s) + (0.2kg)(-3m/s) = 0.9kg.m/s \tag{13}$$

$$p_f = m_1v_{1f} + m_2v_{2f} \tag{14}$$

This equation has two unknowns, therefore, apply the conservation of kinetic energy $E_i = E_f = \frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2$ and conservation of momentum one finds that $v_{1f} = -1.31m/s$ and $v_{2f} = 6.47m/s$ or $v_{1f} = -1.56m/s$ and $v_{2f} = 6.38m/s$. The difference in values is due to the rounding off errors in the numerical calculations of the mathematical quantities. If these two values are averaged the values are $v_{1f} = -1.4m/s$ and $v_{2f} = 6.6m/s$, $p_f = 0.9kg.m/s$. Thus, $p_i = p_f$.

(ii)In the moving frame:

Make use of the Galilean velocity transformation equations. $p'_i = m_1 v'_{1i} + m_2 v'_{2i}$; where $v'_{1i} = v_{1i} - v' = 5m/s - (-2m/s) = 7m/s$. Similarly, $v'_{2i} = -1m/s$ and $p'_i = 1.9kg.m/s$. To find p'_f use $v'_{1f} = v_{1i} - v'$ and $v'_{2f} = v_{2i} - v'$ because the prime system is now moving to the left. Using these results give $p'_f = 1.9kg.m/s$.

Problem 4.

In the derivation of the Galilean transformations that was done in class, we assumed that the motion of the moving frame was just in the positive x direction (i.e. $\vec{v} = v_x\hat{x}$). In this problem, we are going to generalize this to any direction (i.e.$\vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$).

Assuming that at time t = t' = 0 the two frames are together, then at some arbitrary time t later, the distance between the two frames will be $v_x t$ in the x direction (just like the class derivation), $v_y t$ in the y direction and $v_z t$ in the z direction. In vector notation, we write this as

$$\vec{d} = v_x t\hat{x} + v_y t\hat{y} + v_z t\hat{z} = (v_x\hat{x} + v_y\hat{y} + v_z\hat{z})t = \vec{v}t \tag{15}$$

For each direction, the connection between the coordinates in the moving frame (x', y', z', t') and the coordinates in the stationary frame (x, y, z, t) are just

$$x' = x - v_x t \tag{16}$$
$$y' = y - v_y t \tag{17}$$
$$z' = z - v_z t \tag{18}$$
$$t' = t \tag{19}$$

Therefore, in vector notation,

$$\vec{r'} = x'\hat{x} + y'\hat{y} + z'\hat{z} \tag{20}$$
$$\vec{r'} = (x - v_x t)\hat{x} + (y - v_y t)\hat{y} + (z - v_z t)\hat{z} \tag{21}$$
$$\vec{r'} = (x\hat{x} + y\hat{y} + z\hat{z}) - (v_x\hat{x} + v_y\hat{y} + v_z\hat{z})t \tag{22}$$
$$\vec{r'} = \vec{r} - \vec{v}t \tag{23}$$

Problem 5.

The binomial expansion to the next term is $(1+x)^n \cong 1 + nx + 1/2n(n-1)x^2$. In deriving the expected fringe shift, we had the following exact expression,

$$\Delta t = t_1 - t_2 = \frac{2L}{c}[(1 - \frac{v^2}{c^2})^{-1} - (1 - \frac{v^2}{c^2})^{-1/2}] \tag{24}$$

Approximating this using the binomial theorem to the next term we find,

$$\Delta t \cong \frac{2L}{c}[1+\frac{v^2}{c^2}+\frac{v^4}{c^4}-1-\frac{v^2}{2c^2}-\frac{3v^4}{8c^4}] = \frac{Lv^2}{c^3}+\frac{5}{4}\frac{Lv^4}{c^5} \tag{25}$$

The expected fringe is therefore,

$$Shift = \frac{2c\Delta t}{\lambda} \cong 2\frac{Lv^2}{\lambda c^2}+\frac{5}{2}\frac{Lv^4}{\lambda c^4} = 0.4000000055 \tag{26}$$

Where we've plugged the numbers provided in the book (i.e. L = 11 m, v = 30 km/s, λ = 500 nm). Since this correction is much smaller than the resolution of the experiment, we are perfectly justified in keeping only the first term of the binomial expansion.

Problem 6.

If the clocks are spaced 1 million km apart, then the 90^{th} clock down the line is 90 million km from the clock next to me (i.e. $d = 9 \times 10^{10} m$). Light takes $t = d/c = (9 \times 10^{10} m)/(3 \times 10^8 m/s)$ = 300 seconds = 5 minutes to get to me. So the clock will always seem to be 5 minutes behind. In other words, when I see 12 noon on the clock by my side, the light that reaches me from the 90^{th} clock down, must have left 5 minutes prior to that. Hence, the 90^{th} clock down will read 11:55 am.

Problem 7.

(a) $\tau = \gamma\tau'$ where $\beta = \frac{v}{c}$ and

$$\tau = \tau'(1-\beta^2)^{-1/2} = \tau'(1-\frac{v^2}{c^2})^{-1/2} = (2.6\times10^{-8}s)[1-(0.95)^2]^{-1/2} = 8.33\times10^{-8}s \tag{27}$$

(b) $d = v\tau = (0.95)(3 \times 10^8)(8.33 \times 10^8 s) = 24$ m