1. A certain quantum system has only three energy levels, and three corresponding eigenstates. Even if it has nothing to do with rotation and angular momentum, we can identify the states with the states $|m\rangle$ ($m = 1, 0, -1$) of an $\ell = 1$ "effective" angular momentum system, because the angular momentum operators $\vec{L}$ provide us with a convenient set of operators. In terms of these the Hamiltonian for the system is

$$H = \frac{E_0}{\hbar^2}\left(L_x^2 + L_y^2 - 2\hbar L_z\right)$$

(a) Verify that the states $|m\rangle$ are eigenstates of H.

(b) Find the energy eigenvalues of the three states.

(c) The system is initially in the superposition state $\frac{1}{\sqrt{2}}(|1\rangle + |-1\rangle)$, and evolves according to the time-dependent Schrödinger equation. Evaluate the expectation value $\langle L_x \rangle$ as a function of time.

2. In this problem we first consider a system with an arbitrary angular momentum, which is called L but may be orbital or spin (S). From part (b) on we specialize to the angular momentum of a system of two spin $\frac{1}{2}$ particles.

(a) The state vector (wave function) ψ has angular momentum ℓ ($L^2 = \ell(\ell+1)$). New state vectors are generated by acting on ψ with $\vec{L}$, that is $L_i\psi$ and $L_iL_j\psi$, where $i, j = x, y,$ or z. Show that these new state vectors also have angular momentum ℓ. (Hint: Use commutators.)

(b) For the system of two spin $\frac{1}{2}$ particles, show that there is no function of the S-operators ($\vec{S}^{(1)}$, $\vec{S}^{(2)}$ or $\vec{S}$ that will change a singlet state into a triplet state. (You may assume that the general function of the S operators is a polynomial.)

(c) For the same system of two particles, let Q_+ be the operator that changes the singlet state into the $m = 0$ triplet state, and vanishes on all other states:

$$Q_+|0\ 0\rangle = |1\ 0\rangle \qquad Q_+|1\ m\rangle = 0 \quad \text{for } m = -1, 0, +1$$

Is this operator hermitian? If not, find its hermitian adjoint Q_-, defining it by equations like the one above, or as a matrix in the $|s\ m\rangle$ basis.

(d) From what you proved in (c) it follows that $Q_\pm$ cannot be expressed as a function of the S operators. However, Q_-Q_+ and Q_+Q_- can be so expressed. Do so. (One way is first to find the eigenstate for each of these operators that has non-zero eigenvalue; then construct an operator out of the $\vec{S}$'s that vanishes on all states except that eigenstate.)

3. The (time-independent) wave function for the n^{th} energy state of a Hydrogen atom with maximal angular momentum has the (not normalized) form

$$\psi_{n\,n-1\,n-1}(r,\theta,\phi) = r^{n-1}e^{-r/na}\sin^{n-1}\theta\, e^{i(n-1)\phi}$$

where a is the Bohr radius, and the sequence of subscripts on ψ is n, ℓ, m.

(a) What is the most probable value of r in this state? Compare with the value $r = an^2$ given by the simple Bohr model. (Hint: first you must figure out the probability that the electron would be found between r and $r + dr$.)

(b) For the corresponding time-dependent state $\Psi(r,\theta,\phi,t)$, show that the probability density, and hence the expectation value of all operators, is constant in time. The energy of these states is $E_n = E_1/n^2$.

(c) Time-dependent probability density occurs for superposition states of *time-dependent* stationary states. Let $\Psi = \Psi_{n\,n-1\,n-1} + \Psi_{n+1\,n\,n}$ and show that the probability density consists of time-independent terms as in (b) plus time-dependent "interference terms." Using the above expression for ψ and its time dependence, and in the limit of large n, write this interference term explicitly as a function of r, θ, ϕ and t. (Do not worry about normalization.) Show that it really depends only on a linear combination $\phi - \omega t$, and find ω. This means that the probability density rotates rigidly with angular frequency ω. Compare with the Bohr theory value of this frequency, $\hbar/ma^2n^3$, with which the electron rotates about the nucleus.

Formulas

Time-dependent Schrödinger equation	$i\hbar\frac{\partial\psi(x,t)}{\partial t} = H\psi(x)$
Time-independent Schrödinger equation	$H\psi(x) = E\psi(x)$
Hamiltonian operator	$H = \frac{p^2}{2m} + V(x)$
Momentum operator	$p = \frac{\hbar}{i}\frac{\partial}{\partial x}$
Expansion	$\lvert\psi\rangle = \Sigma\,\lvert e_n\rangle\langle e_n\vert\psi\rangle$
Commutator	$[p, x] = \frac{\hbar}{i}$

Expectation value $\langle Q \rangle = \langle \psi | Q | \psi \rangle$

Matrix elements $Q_{mn} = \langle e_m | Q | e_n \rangle$

Allowed energies of H atom $E_n = -\left[\frac{m}{2\hbar^2} \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 \right] \frac{1}{n^2}$

Angular momentum $\vec{L} = \vec{r} \times \vec{p} \quad [L_x, L_y] = i\hbar L_z;\ [L_y, L_z] = i\hbar L_x;\ [L_z, L_x] = i\hbar L_y;\ [L^2, L_i] = 0$

Singlet state $|0\,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$

Triplet state $|1\,1\rangle = \uparrow\uparrow \quad |1\,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) \quad |1\ {-1}\rangle = \downarrow\downarrow$