HW 9 - SOLUTION

PHY 402 - FALL 2014

I. The relevant matrix element is:

$$\langle n | \hat{x}^4 | 0 \rangle = \langle n | \left[\sqrt{\frac{\hbar}{2m\omega}} (\hat{a}_+ + \hat{a}_-) \right]^4 | 0 \rangle$$

$$= \left(\frac{\hbar}{2m\omega}\right)^2 \langle n | (\hat{a}_+ + \hat{a}_-)^3 \underbrace{(\hat{a}_+ + \hat{a}_-) | 0 \rangle}_{|1\rangle}$$

$$= \left(\frac{\hbar}{2m\omega}\right)^2 \langle n | (\hat{a}_+ + \hat{a}_-)^2 \left[\sqrt{2}\, |2\rangle + |0\rangle \right]$$

$$= \left(\frac{\hbar}{2m\omega}\right)^2 \langle n | (\hat{a}_+ + \hat{a}_-) \left[\sqrt{6}\, |3\rangle + 2\,|1\rangle + |1\rangle \right]$$

$$= \left(\frac{\hbar}{2m\omega}\right)^2 \langle n | \left[\sqrt{24}\, |4\rangle + \sqrt{18}\, |2\rangle + 2\sqrt{2}\, |2\rangle + 2\,|0\rangle + \sqrt{2}\,|2\rangle + |0\rangle \right]$$

$$= \left(\frac{\hbar}{2m\omega}\right)^2 \left[\sqrt{24}\, \delta_{n4} + 3\sqrt{2}\, \delta_{n2} + 3\, \delta_{n0} \right]$$

$$c_n^{(1)} = \frac{1}{\hbar^2} e^{-i \frac{\hbar\omega}{2} t/\hbar} \int_0^t dt' \, \lambda \left(\frac{\hbar}{2m\omega}\right)^2 \begin{cases} e^{-i2\hbar\omega t'/\hbar} \; \overset{6}{\boxed{3}}\sqrt{2}, & n=2 \\ e^{-i4\hbar\omega t'/\hbar} \; \sqrt{24}, & n=4 \end{cases}$$

$$|c_n^{(1)}|^2 = \frac{1}{\hbar^2} \lambda^2 \left(\frac{\hbar}{2m\omega}\right)^4 \begin{cases} \left| \frac{e^{-i2\omega t} - 1}{2\omega} \right|^2 \overset{72}{\boxed{18}}, & n=2 \\ \left| \frac{e^{-i4\omega t} - 1}{4\omega} \right|^4 24, & n=4 \end{cases}$$

$$= \begin{cases} \dfrac{\lambda^2 \hbar^2}{16 m^4 \omega^4} \dfrac{1}{4\omega^2}\, 4 \sin^2 2\omega t \; \overset{72}{\boxed{18}} \, , & n=2 \\[2ex] \dfrac{\lambda^2 \hbar^2}{16 m^4 \omega^4} \dfrac{1}{16\omega^2}\, 4 \sin^2 4\omega t \; 24 \, , & n=4 \end{cases}$$

Possible outcomes are $\frac{5\hbar\omega}{2}$ and $\frac{9\hbar\omega}{2}$ w/ probabilities

$$\frac{\lambda^2 \hbar^2}{m^4 \omega^6} \begin{cases} 18/16 & \text{for } 5\hbar\omega/2 \\ 6/16 & \text{for } 9\hbar\omega/2 \end{cases}$$

There's no contradiction ~~the~~ with conservation of energy since work has to be done on the system by whoever/whatever is turning the perturbation on and off.

II. There are three first excited states:

$$|\text{ground}\rangle = |\underset{n_x^1}{1}\,\underset{n_y^1}{1}\,\underset{n_z^1}{1},\ \underset{n_x^2}{1}\,\underset{n_y^2}{1}\,\underset{n_z^2}{1}\rangle$$

$$|1, A\rangle = \frac{|211, 111\rangle + |111, 211\rangle}{\sqrt{2}}$$

$$|1, B\rangle = \frac{|121, 111\rangle + |111, 121\rangle}{\sqrt{2}}$$

$$|1, C\rangle = \frac{|112, 111\rangle + |111, 112\rangle}{\sqrt{2}}$$

symmetric under the exchange of the two particles

The relevant matrix elements are: $\langle 1, A | \delta(\hat{r}_1 - \hat{r}_2) | \text{ground} \rangle$,

$\langle 1, B | \delta(\hat{r}_1 - \hat{r}_2) | \text{ground} \rangle$

and

$\langle 1, C | \delta(\hat{r}_1 - \hat{r}_2) | \text{ground} \rangle$

Let's look at a building block needed for all these matrix elements:

$$\langle 112, 111 | \delta(\hat{r}_1 - \hat{r}_2) | 111, 111 \rangle = \int d^3r_1\, d^3r_2\, \langle 112, 111 | \underbrace{\delta(\hat{r}_1 - \hat{r}_2) | r_1 r_2 \rangle}_{\delta(r_1 - r_2) | r_1 r_2 \rangle} \langle r_1 r_2 | 111, 111 \rangle$$

$$= \int d^3r\, \langle 112, 111 | r\, r \rangle \langle r\, r | 111, 111 \rangle$$

$$= \int d^3r\, \langle 1|x\rangle \langle 1|y\rangle \langle 2|z\rangle \langle 1|x\rangle \langle 1|y\rangle \langle 1|z\rangle \quad \langle x|1\rangle \langle y|1\rangle \langle z|1\rangle \langle x|1\rangle \langle y|1\rangle \langle z|1\rangle$$

one-dimensional, one particle states)

$$= \int_0^L dx\, |\langle x|1\rangle|^4 \int_0^L dy\, |\langle y|1\rangle|^4 \underbrace{\int_0^L dz\, \langle 2|z\rangle \langle 1|z\rangle \langle z|1\rangle^2}_{\underbrace{\int_0^L dz \left(\frac{2}{L}\right)^2 \sin^3\frac{z\pi}{L} \sin\frac{2\pi z}{L}}_{=0}}$$

$= 0$

Similarly

$$\left[\frac{\langle 112, 111| + \langle 111, 112|}{\sqrt{2}}\right] \delta(\hat{r}_1 - \hat{r}_2) | 111, 111 \rangle = 0$$

There's nothing special to $|1,A\rangle$ as compared to $|1,B\rangle$ and $|1,C\rangle$ so the matrix elements $\langle 1,B|\,\delta(\vec{r}_1-\vec{r}_2)\,|\text{ground}\rangle$ and $\langle 1,C|\,\delta(\vec{r}_1-\vec{r}_2)\,|\text{ground}\rangle$ should vanish too. A more formal way to argue is to observe that the states $|1,B\rangle$ and $|1,C\rangle$ are obtained from $|1,A\rangle$ by a $\pi/2$ rotation. But the operator $\delta(\vec{r}_1-\vec{r}_2)$ is invariant under these rotations and so is the ground state, so all these matrix elements should be the same.