PHY 402 - Fall 2014
homework 8 solutions

I. state at time of measurement $= |\uparrow\rangle$ (eigenstate of $\hat{S}_z$ w/ eigenvalue $+\hbar/2$)

operator corresponding to the observable being measured $= \vec{n}\cdot\hat{\vec{S}} = \frac{\sqrt{3}}{2}\hat{S}_x + \frac{1}{2}\hat{S}_z$

$$\xrightarrow[\text{in the } |\uparrow\rangle, |\downarrow\rangle \text{ basis}]{} \frac{\hbar}{2}\begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} \equiv S_{z'}$$

eigenstates / eigenvalues of $\hat{S}_{z'}$:

$$\underbrace{\frac{\hbar}{2}\frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}}_{S_{z'}} \underbrace{\frac{\sqrt{3}}{2}\begin{pmatrix} -1/\sqrt{3} \\ 1 \end{pmatrix}}_{\text{eigenvector}} = \frac{\hbar}{2}\frac{1}{2}\frac{\sqrt{3}}{2}\begin{pmatrix} -\frac{1}{\sqrt{3}}+\sqrt{3} \\ -1-1 \end{pmatrix} = \frac{\hbar}{2}\frac{1}{2}\begin{pmatrix} 2/\sqrt{3} \\ -2 \end{pmatrix}$$

$$= \underbrace{-\frac{\hbar}{2}}_{\text{eigenvalue}} \underbrace{\begin{pmatrix} -1/\sqrt{3} \\ 1 \end{pmatrix}}_{\text{eigenvector}}$$

$$\overbrace{\frac{\hbar}{2}\frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}} \overbrace{\frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}} = \frac{\hbar}{2}\frac{1}{4}\begin{pmatrix} 2\sqrt{3} \\ 2 \end{pmatrix} = \underbrace{\frac{\hbar}{2}}_{\text{eigenvalue}} \underbrace{\frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}}_{\text{eigenvector}}$$

$\Downarrow$

eigenvectors of $\hat{S}_{z'}$ are

$$|z'+\rangle = \frac{\sqrt{3}}{2}|\uparrow\rangle + \frac{1}{2}|\downarrow\rangle \quad (\text{eigenvalue } \hbar/2)$$

and

$$|z'-\rangle = -\frac{1}{2}|\uparrow\rangle + \frac{\sqrt{3}}{2}|\downarrow\rangle \quad (\text{eigenvalue } -\hbar/2)$$

expand initial state in the $|z',+\rangle$, $|z',-\rangle$ basis

$$|\uparrow\rangle = \alpha\, |z',+\rangle + \beta\, |z',-\rangle = \frac{\sqrt{3}}{2}\, |z',+\rangle - \frac{1}{2}\, |z',-\rangle$$

$$\langle z',+|\uparrow\rangle = \left[\frac{\sqrt{3}}{2}\langle\uparrow| + \frac{1}{2}\langle\downarrow|\right]|\uparrow\rangle = \frac{\sqrt{3}}{2}$$

$$\langle z',-|\uparrow\rangle = \left[-\frac{1}{2}\langle\uparrow| + \frac{\sqrt{3}}{2}\langle\downarrow|\right]|\uparrow\rangle = -\frac{1}{2}$$

prob. of $S_{z'} = \hbar/2$ $= |\alpha|^2 = 3/4$

prob. of $S_{z'} = -\hbar/2$ $= |\beta|^2 = 1/4$

III. turning points : $\dfrac{\lambda}{\cosh^2 \bar{x}/L} = E \Rightarrow \bar{x} = \pm L \operatorname{arccosh}\sqrt{\dfrac{\lambda}{E}}$

$$J(E) \equiv \int_{-\bar{x}}^{\bar{x}} dx \sqrt{2m\left(E - \frac{\lambda}{\cosh^2 x/L}\right)} = \left(n+\tfrac{1}{2}\right)\pi\hbar$$

$$\frac{dJ}{dE} = \sqrt{2M} \int_{-\bar{x}}^{\bar{x}} dx \frac{1}{2\sqrt{E - \frac{\lambda}{\cosh^2 x/L}}}$$

(boundary terms vanish since the integrand vanishes at $x = \pm\bar{x}$)

using $y = \dfrac{\sinh x/L}{\sqrt{\lambda/E - 1}}$ we have

$$\frac{dJ}{dE} = \sqrt{2M}\, L \int_0 dy \frac{1}{\sqrt{(1+y^2)E - \lambda}} = \pi L \sqrt{\frac{M}{-2E}}$$

$$J(E) = \int_\lambda^E dE' \; \pi L \sqrt{\frac{M}{-2E'}} + \underbrace{J(E=\lambda)}_{=0}$$

$$= \pi \sqrt{2ML^2} \left(\sqrt{-\lambda} - \sqrt{-E}\right)$$

$$J(E) = \left(n+\tfrac{1}{2}\right)\pi\hbar \;\Rightarrow\; E = -\frac{\hbar^2}{2ML^2}\left[\frac{\sqrt{-2\lambda M L^2}}{\hbar} - \frac{1}{2} - n\right]^2$$

Setting $n = 1, 2, \ldots 10$ we have the WKB approximation to the energy levels

II. Ansatz : $\psi(r,\theta,\phi) = A\, e^{-\alpha r}$ (I won't keep the numerical factors for this estimate)

$$1 = \int d^3r\, |\psi|^2 \sim A^2 \int_0^\infty dr\, r^2 e^{-\alpha r 2} \sim A^2 \frac{1}{\alpha^3} \Rightarrow A \sim \alpha^{3/2}$$

$$E_\alpha \sim \int d^3r\, \psi^+(r)\left[-\frac{\hbar^2}{2M}\nabla^2 + \sigma r\right]\psi(r)$$

$$\sim \int_0^\infty dr\, r^2 \alpha^3 \left[\left(\frac{\partial}{\partial r} e^{-\alpha r}\right)^2 \frac{\hbar^2}{2M} + \sigma r\right] e^{-\alpha r}$$

$$\sim \frac{\hbar^2}{M}\alpha^3 \frac{1}{\alpha} + \sigma\frac{\alpha^3}{\alpha^4} \sim \frac{\hbar^2}{M}\alpha^2 + \frac{\sigma}{\alpha}$$

$$\frac{dE_\alpha}{d\alpha} = 0 \Rightarrow \frac{\hbar^2}{M}\alpha - \frac{\sigma}{\alpha^2} \Rightarrow \alpha \sim \left(\frac{M\sigma}{\hbar^2}\right)^{1/3}, \quad E \sim \frac{\hbar^2}{M}\left(\frac{M\sigma}{\hbar^2}\right)^{2/3} + \sigma\left(\frac{\hbar^2}{M\sigma}\right)^{1/3}$$

$$\sim \frac{\hbar^{2/3}\sigma^{2/3}}{M^{1/3}} \sim \left(\frac{\hbar^2\sigma^2}{M}\right)^{1/3}$$

$$\boxed{E \sim \left(\frac{\hbar^2\sigma^2}{M}\right)^{1/3} \xrightarrow[M\to\infty]{} 0}$$

ii) $$E \sim \left(\frac{\hbar^2 \sigma^2}{M_{charm}}\right)^{1/3} \sim \left[\frac{\left(6\times10^{-16}\ \text{eV s}\right)^2}{1.3\times10^{9}\ \frac{\text{eV}}{c^2}}\left(\frac{9\times10^{8}\ \text{eV}}{10^{-15}\ \text{m}}\right)^2\right]^{1/3}$$

speed of light

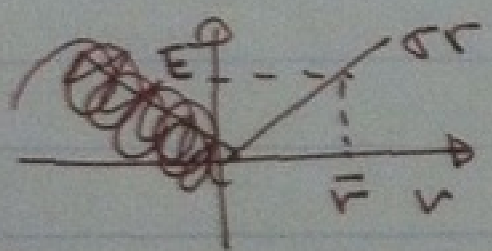
$$\sim \left[\frac{\left(6\times10^{-16}\right)^2\left(9\times10^{23}\right)^2}{1.3\times10^{9}}\ \text{eV}^3\ \frac{s^2}{m^2}\left(3\times10^{8}\right)^2\frac{m^2}{s^2}\right]^{1/3}$$

$\sim 3 \times 10^{8}$ eV

Actually, just a factor of ~4, not that non-relativistic!

$E \ll M_{charm} c^2 \Rightarrow$ non-relativistic system

The bottom quark mass is even greater so bottomium is even more non-relativistic.

iii) 

$$E = \sigma \bar{r} \Rightarrow \bar{r} = \frac{E}{\sigma}$$

Twice the reduced mass: $M = 2\mu_{red}$

$$\underbrace{2\int_0^{\bar{r}} dr \sqrt{M(E - \sigma r)}}_{2\sqrt{M}\ \frac{2}{3\sigma} E^{3/2}} = n\pi\hbar$$

whole period goes from 0 to $\bar{r}$ twice.

$\Downarrow$

$$E_n = \left(\frac{3\sigma}{4}\frac{\pi\hbar n}{\sqrt{M}}\right)^{2/3} = \left(\frac{3\pi}{4}\right)^{2/3}\left(\frac{\hbar^2\sigma^2}{M}\right)^{1/3} n^{2/3}$$

iv) This result above becomes more precise for highly excited states with $n \gg 1$ whose wavefunction changes over a short distance.