PHY 402 - HW 6 SOLUTION
FALL 2014

A. First order pert. theory gives the corrections we need to order $O(\lambda)$:

$$\hat{H} = \underbrace{\frac{\hat{p}^2}{2M} + \frac{M\omega^2}{2}\hat{x}^2}_{\hat{H}_0} + \underbrace{\lambda\,\hat{x}^4}_{\lambda\hat{H}_1} \quad \Rightarrow \quad \lambda E_0^{(1)} = \langle 0|\hat{H}_1|0\rangle$$

unperturbed states; will drop superscript "(0)" for now

$$\lambda\,|0\rangle^{(1)} = \sum_{m\neq 0} \frac{\langle m|\hat{H}_1|0\rangle}{E_0^{(0)} - E_m^{(0)}}\,|m\rangle$$

We need to compute $\langle m|\hat{x}^4|0\rangle$:

$$\langle m|\hat{x}^4|0\rangle = \sqrt{\frac{\hbar}{2M\omega}}^{\,4}\,\langle m|(\hat{a}_+ + \hat{a}_-)^4|0\rangle \qquad \hat{a}_\pm = \frac{M\omega\hat{x} \mp i\hat{p}}{\sqrt{2\hbar M\omega}}$$

$$= \left(\frac{\hbar}{2M\omega}\right)^2 \underbrace{\langle m|(\hat{a}_+ + \hat{a}_-)}_{\sqrt{m+1}\,\langle m+1| + \sqrt{m}\,\langle m-1|}(\hat{a}_+ + \hat{a}_-)^2\underbrace{(\hat{a}_+ + \hat{a}_-)|0\rangle}_{|1\rangle}$$

$$= \left(\frac{\hbar}{2M\omega}\right)^2\Big[\sqrt{m+1}\,\langle m+1|\,\hat{a}_+^2 + \hat{a}_-^2 + \hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+\,|1\rangle$$
$$+ \sqrt{m}\,\langle m-1|\,\hat{a}_+^2 + \hat{a}_-^2 + \hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+\,|1\rangle\Big]$$

$$= \left(\frac{\hbar}{2M\omega}\right)^2 \sqrt{m+1}\,\Big[\langle m+1|\sqrt{6}\,|3\rangle + \langle m+1|1\rangle + \sqrt{2}^2\langle m+1|1\rangle\Big]$$
$$+ \left(\frac{\hbar}{2M\omega}\right)^2 \sqrt{m}\,\Big[\langle m-1|\sqrt{6}\,|3\rangle + \langle m-1|1\rangle + \sqrt{2}^2\langle m-1|1\rangle\Big]$$

$$= \left(\frac{\hbar}{2M\omega}\right)^2\Big[\underbrace{\sqrt{6}\sqrt{m+1}\,\delta_{m2}}_{\sqrt{18}\,\delta_{m2}} + \underbrace{\sqrt{m+1}\,3\,\delta_{m0}}_{3\,\delta_{m0}} + \underbrace{\sqrt{m}\sqrt{6}\,\delta_{m4}}_{\sqrt{24}\,\delta_{m4}} + 3\underbrace{\sqrt{m}}_{3\sqrt{2}}\delta_{m2}\Big]$$

$$= \left(\frac{\hbar}{2M\omega}\right)^2\Big[3\,\delta_{m0} + 6\sqrt{2}\,\delta_{m2} + \sqrt{24}\,\delta_{m4}\Big]$$

writting the superscipts "(0)" again:

so $$E_0 \simeq E_0^{(0)} + \underbrace{\lambda E_0^{(1)}}_{{}^{(0)}\langle 0|\hat{H}_1|0\rangle^{(0)}} = \frac{\hbar\omega}{2} + 3\lambda\left(\frac{\hbar}{2M\omega}\right)^2$$

$$\underset{\substack{\uparrow \\ \text{ground state} \\ \text{of } \hat{H}}}{|0\rangle} \simeq |0\rangle^{(0)} + \sum_{m\neq 0} \frac{{}^{(0)}\langle m|\hat{H}_1|0\rangle^{(0)}}{\hbar\omega\left(\frac{1}{2}-m-\frac{1}{2}\right)} |0\rangle^{(0)}$$

$$= |0\rangle^{(0)} - \frac{\lambda}{\hbar\omega m}\left[6\sqrt{2}\,|2\rangle^{(0)} + \sqrt{24}\,|4\rangle^{(0)}\right]$$

B. i) Again, first order pert. theory will give the energies up to $\mathcal{O}(V_0)$:

$$\underbrace{\Delta E_n}_{\text{shift in energy up to order } \mathcal{O}(V_0)} = {}^{(0)}\langle n|\hat{V}|n\rangle^{(0)} = \int_0^L dx\,dy\; {}^{(0)}\langle n|x\rangle\langle x|\hat{V}|y\rangle\langle y|n\rangle^{(0)}$$

$$= \int_0^L dx \int_0^L dy \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\delta(x-y)V_0\,\Theta\left(\frac{L}{2}-x\right)\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi y}{L}\right)$$

Θ ← step function

$$= V_0\int_0^{L/2} dx\, \frac{2}{L}\sin^2\left(\frac{n\pi x}{L}\right)$$

$$= \frac{V_0 L}{4}\frac{2}{L} = \frac{V_0}{2}$$

$$\boxed{E_n \simeq \frac{\pi^2\hbar^2 n^2}{2ML^2} + \frac{V_0}{2}}$$

ii) $\frac{V_0}{2} \ll \frac{\pi^2\hbar^2}{2ML^2}n^2 \Rightarrow V_0 \ll \frac{\pi^2\hbar^2}{ML^2}n^2$. Pert. theory works better for highly excited states.