QUANTUM PHYSICS II

PROBLEM SET 4

due October 7, before class

A. Center-of-mass separation

Frequently the interaction potential between two particles depends only on the distance between them, that is $V(\mathbf{r}_1, \mathbf{r}_2) = V(\mathbf{r}_1 - \mathbf{r}_2)$. In fact, this follows from translation invariance of the laws of Physics if one assumes V does depend on $\hat{p}$ (what sometimes happens). Anyway, the point of this problem is to write the time-independent Schrödinger equation in position space in terms of the relative $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and center-of-mass $\mathbf{R} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/(m_1 + m_2)$ coordinates.

i) Show that $\mathbf{r}_1 = \mathbf{R} + \mu\mathbf{r}/m_1, \mathbf{r}_2 = \mathbf{R} - \mu\mathbf{r}/m_2$ and $\nabla_1 = (\mu/m_2)\nabla_\mathbf{R} + \nabla_\mathbf{r}, \nabla_2 = (\mu/m_1)\nabla_\mathbf{R} - \nabla_\mathbf{r}$ where

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \tag{1}$$

is the reduced mass.

ii) Show that the time-independent Schr odinger equation in position space becomes

$$\left[-\frac{\hbar^2}{2(m_1+m_2)}\nabla^2_\mathbf{R} - \frac{\hbar^2}{2\mu}\nabla^2_\mathbf{r} + V(\mathbf{r})\right]\psi = E\psi, \tag{2}$$

where

$$\nabla^2_\mathbf{R} = \frac{\partial^2}{\partial R_x^2} + \frac{\partial^2}{\partial R_y^2} + \frac{\partial^2}{\partial R_z^2} \tag{3}$$

and similarly for $\nabla^2_\mathbf{r}$.

iii) Separate variables by trying solutions of the form $\psi(\mathbf{R}, \mathbf{r}) = \psi_\mathbf{R}(\mathbf{R})\psi_\mathbf{r}(\mathbf{r})$. Note that $\psi_\mathbf{R}$ satisfies the one-particle Schrödinger equation with total mass $m_1 + m_2$ and no potential energy while $\psi_\mathbf{r}$ satisfies the same equation with potential V and mass μ and the total energy $E = E_\mathbf{R} + E_\mathbf{r}$. This means that the center-of-mass moves like a free particles and the relative position behaves like a *single* particle with a reduces mass.

iv) Chlorine has two isotopes Cl^{35} and Cl^{37}. They both occur naturally so, in a sample of HCl molecules some will have a Cl^{35} atom, other will have a Cl^{37} atoms. Show that the vibrational spectrum of HCl should consist of closed spaced states with a splitting given by $\Delta\nu = 7.5 \times 10^{-4}\nu$, where ν is the frequency of the emitted photon. Hint: Think of the vibration of the molecule as a harmonic oscillator with angular frequency $\omega = \sqrt{k/\mu}$, μ is the reduced mass and k (presumably) is pretty much the same for both kinds of molecules.

B. When the intuition needs help, make a plot

Write down the wave function in position space of the ground state of two spin 1/2 fermions in an infinite square potential of side L *assuming both spins are up*. Make a plot of the wave function as a function of x_1 and x_2. What is the set of points in the (x_1, x_2) plane where the wave function vanishes?

C. Position basis

i) In the space of one dimensional wave functions, compute the matrix elements of the operator $\hat{x}$ in the eigenfunctions of position.

ii) In the space of one dimensional wave functions, compute the matrix elements of the operator $\hat{p}$ in the eigenfunctions of position.

D. Momentum basis

i) In the space of one dimensional wave functions, compute the matrix elements of the operator $\hat{x}$ in the eigenfunctions of momentum.

ii) In the space of one dimensional wave functions, compute the matrix elements of the operator $\hat{p}$ in the eigenfunctions of momentum.

E. Particles in a harmonic well

Three electrons are put in the same harmonic oscillator well. Neglecting the interaction among the electrons, write down the wave function in position space of the ground state of the system (including orbital and spin variables).

F. EPR argument

State the essence of the EPR argument in a small paragraph.