

MIDTERM SOLUTION - PHY 402 - Fall 12

PROBLEM I

$$1) \text{ Since } \hat{H} |a\rangle = \frac{E_0}{2} |a\rangle + \frac{\sqrt{3} E_0}{2} |b\rangle$$

$$\hat{H} |b\rangle = -\frac{E_0}{2} |b\rangle + \frac{\sqrt{3} E_0}{2} |a\rangle$$

$$\begin{aligned}\hat{H} \left(\frac{\sqrt{3}|a\rangle + |b\rangle}{2} \right) &= \frac{\sqrt{3}}{2} \left(\frac{E_0}{2} |a\rangle + \frac{\sqrt{3} E_0}{2} |b\rangle \right) + \frac{1}{2} \left(-\frac{E_0}{2} |b\rangle + \frac{\sqrt{3} E_0}{2} |a\rangle \right) \\ &= E_0 \left[\frac{\sqrt{3}}{2} |a\rangle + \frac{1}{2} |b\rangle \right]\end{aligned}$$

$$\begin{aligned}\hat{H} \left(\frac{|a\rangle - \sqrt{3}|b\rangle}{2} \right) &= \frac{1}{2} \left(\frac{E_0}{2} |a\rangle + \frac{\sqrt{3} E_0}{2} |b\rangle \right) - \frac{\sqrt{3}}{2} \left(-\frac{E_0}{2} |b\rangle + \frac{\sqrt{3} E_0}{2} |a\rangle \right) \\ &= -E_0 \left(\frac{|a\rangle - \sqrt{3}|b\rangle}{2} \right)\end{aligned}$$

Q&A

2) Since $\hat{H} \left(\frac{\sqrt{3}|a\rangle + |b\rangle}{2} \right) = E_0 \left(\frac{\sqrt{3}|a\rangle + |b\rangle}{2} \right)$, $\frac{\sqrt{3}|a\rangle + |b\rangle}{2}$ is an eigenvector of \hat{H} w/ eigenvalue E_0 .

Similarly, $\frac{|a\rangle - \sqrt{3}|b\rangle}{2}$ is an eigenvector w/ eigenvalue $-E_0$.

$E_0, -E_0$

3) Using the eigenstates of H $|1\rangle = \frac{\sqrt{3}|a\rangle + |b\rangle}{2}$, $|2\rangle = \frac{|a\rangle - \sqrt{3}|b\rangle}{2}$,
The initial condition can be written as

$|\Psi(0)\rangle = |a\rangle = |1\rangle |1\rangle + |2\rangle |2\rangle = \frac{\sqrt{3}}{2} |1\rangle + \frac{1}{2} |2\rangle$. The
energy eigenstates $(1, 2)$ evolve in time as $|1, 2, t\rangle = e^{-iE_{1,2}t/\hbar} |1, 2\rangle$
so, since the Schrödinger eq. is linear

$$|\Psi(t)\rangle = \frac{\sqrt{3}}{2} e^{-iE_1 t/\hbar} |1\rangle + \frac{1}{2} e^{+iE_2 t/\hbar} |2\rangle = \xrightarrow{\quad}$$

(2)

$$|\psi(t)\rangle = e^{-iE_0 t \hbar} \frac{\sqrt{3}}{2} \left(\frac{1}{2}|1\rangle + \frac{1}{2}|5\rangle \right) + e^{iE_0 t \hbar} \frac{1}{2} \left(\frac{1}{2}|1\rangle - \frac{\sqrt{3}}{2}|5\rangle \right)$$

$$= \left(\frac{3}{4} e^{-iE_0 t \hbar} + \frac{1}{4} e^{iE_0 t \hbar} \right) |1\rangle + \left(\frac{1}{4} e^{-iE_0 t \hbar} - \frac{\sqrt{3}}{4} e^{iE_0 t \hbar} \right) |5\rangle$$

4) $|\psi(t)\rangle = \underbrace{e^{-iE_0 t \hbar} \frac{\sqrt{3}}{2}}_{\text{prob. ampl. of } E_0} |1\rangle + \underbrace{\frac{1}{2} e^{iE_0 t \hbar}}_{\substack{\text{prob. ampl.} \\ \text{of } -E_0}} |5\rangle$

eigenstates
of \hat{A}

probability of E_0 : $|e^{-iE_0 t \hbar} \frac{\sqrt{3}}{2}|^2 = \frac{3}{4}$

probability of $-E_0$: $|e^{iE_0 t \hbar} \frac{1}{2}|^2 = \frac{1}{4}$

5) After the measurement the wave function collapses to the ground state $|2\rangle$ (assuming $E_0 > 0$, otherwise the ground state would be $|1\rangle$). That state evolves just by a phase $|\psi(t)\rangle = e^{iE_0 t \hbar} |2\rangle$ and never acquires a component along $|1\rangle$. Any further measurement of the energy will give the ground state energy w/ 100% probability

prob. of ground state = 100%

(3)

PROBLEM II.

6) ~~Take the ground~~ The energy of the system will be the sum of the energy of each particle. Thus, in the ground state, both particles will be in the single-particle ground state. That makes the orbital part of the wave function symmetric under the exchange of particle positions and, since the particles are fermions, the spin part must be anti-symmetric. The anti-symmetric combination of spin ψ_2 has total spin $S=0$:

$$\frac{|\psi_2 - \psi_2\rangle - |\psi_2 \psi_2\rangle}{\sqrt{2}} = |S=0, m_s=0\rangle$$

Spin $S=0$

$$7) \hat{S}^2 = (\hat{S}_1 + \hat{S}_2)^2 = \hat{S}_1^2 + \hat{S}_2^2 + 2 \hat{S}_1 \cdot \hat{S}_2 \Leftrightarrow \hat{S}_1 \cdot \hat{S}_2 = \frac{\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2}{2}$$

$$\langle S=0, m=0 | \hat{S}_1 \cdot \hat{S}_2 | S=0, m=0 \rangle = \langle S=0, m=0 | \frac{\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2}{2} | S=0, m=0 \rangle$$

$$= \langle S=0, m=0 | \left[\frac{0(0+1) - \frac{1}{2}(\frac{1}{2}+1) - \frac{1}{2}(\frac{1}{2}+1)}{2} \right] | S=0, m=0 \rangle$$

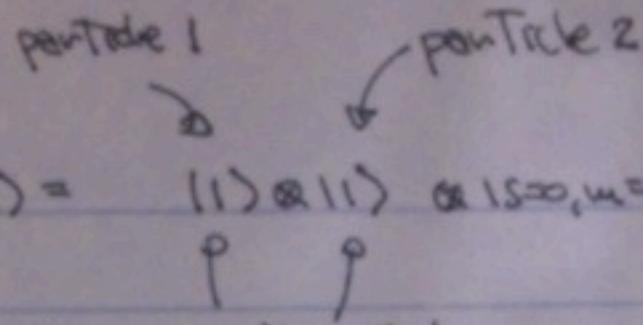
$\Rightarrow \boxed{\langle S=0, m=0 | \hat{S}_1 \cdot \hat{S}_2 | S=0, m=0 \rangle = -\frac{3}{4}}$

$$\langle S=1, m=1 | \hat{S}_1 \cdot \hat{S}_2 | S=1, m=1 \rangle = \langle S=1, m=1 | \frac{\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2}{2} | S=1, m=1 \rangle$$

$$= \langle S=1, m=1 | \left[\frac{1(1+1) - \frac{1}{2}(\frac{1}{2}+1) - \frac{1}{2}(\frac{1}{2}+1)}{2} \right] | S=1, m=1 \rangle$$

$\Rightarrow \boxed{\langle S=1, m=1 | \hat{S}_1 \cdot \hat{S}_2 | S=1, m=1 \rangle = \frac{1}{4}}$

(4)



$$\text{ground state : } |g_{\text{ground}}\rangle = |1\rangle \otimes |1\rangle \text{ or } |s=0, m=0\rangle$$

single particle
ground state

The shift in energy is given by 1st order pert. theory as

$$\Delta E = \langle g_{\text{ground}} | -g \hat{\vec{s}}_1 \cdot \hat{\vec{s}}_2 \int dx \langle x|x| \rangle |g_{\text{ground}}\rangle$$

$$= -g \underbrace{\langle s=0, m=0 | \hat{\vec{s}}_1 \cdot \hat{\vec{s}}_2 | s=0, m=0 \rangle}_{-\frac{3}{4}} \langle 11 \otimes 11 | \int dx \langle x|x| \rangle |11 \otimes 11 \rangle$$

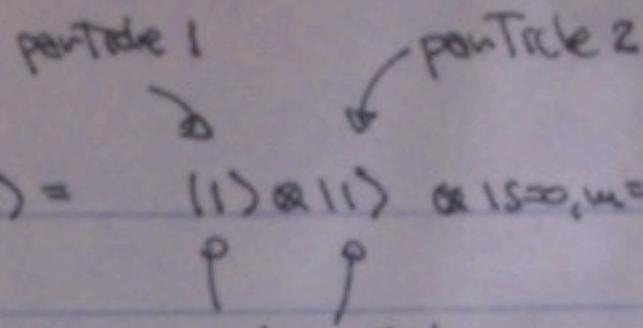
$$= \frac{3g}{4} \underbrace{\int dx \langle 11x \rangle \langle x11 \rangle}_{\langle (x11) \rangle^2}$$

$$= \frac{3g}{4} \int_0^L dx \left[\sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) \right]^4 = \frac{3g}{4} \frac{4}{L^2} \frac{3L}{8} \boxed{= \frac{9}{8} \frac{g}{L}}$$

9) For any state w/ total spin 1 (symmetric in spin), the orbital part must be anti-symmetric under the exchange of particles. Thus they have the form $| \text{excited} \rangle = \frac{|1a\rangle \otimes |1b\rangle - |1b\rangle \otimes |1a\rangle}{\sqrt{2}}$ or $\langle s=1, m=1 \rangle$, for some $|1a\rangle$ and $|1b\rangle$. The 1^1S excited state will have $\sqrt{2}$ have $|1a\rangle = |1\rangle$ and $|1b\rangle = |2\rangle$. Computing the perturbation as above we have

$$\begin{aligned} \langle \text{excited} | -g \hat{\vec{s}}_1 \cdot \hat{\vec{s}}_2 | \text{excited} \rangle &= -g \langle s=1, m=1 | \hat{\vec{s}}_1 \cdot \hat{\vec{s}}_2 | s=1, m=1 \rangle \\ &\quad \left[\frac{\langle 11 \otimes 21 - 21 \otimes 11 \rangle}{\sqrt{2}} \right] \int dx \langle x|x| \rangle |11 \otimes 11 \rangle \\ &\quad \left[\frac{\langle 11 \otimes 12 - 12 \otimes 11 \rangle}{\sqrt{2}} \right] \end{aligned}$$

(4)



$$\text{ground state : } |g_{\text{ground}}\rangle = |1\rangle \otimes |1\rangle \text{ or } |s=0, m=0\rangle$$

single particle
ground state

The shift in energy is given by 1st order pert. theory as

$$\Delta E = \langle g_{\text{ground}} | -g \hat{\vec{s}}_1 \cdot \hat{\vec{s}}_2 \int dx \langle x|x| \rangle |g_{\text{ground}}\rangle$$

$$= -g \underbrace{\langle s=0, m=0 | \hat{\vec{s}}_1 \cdot \hat{\vec{s}}_2 | s=0, m=0 \rangle}_{-\frac{3}{4}} \langle 11 \otimes 11 | \int dx \langle x|x| \rangle |11 \otimes 11 \rangle$$

$$= \frac{-3g}{4} \underbrace{\int dx \langle 11x \rangle \langle x11 \rangle}_{\langle (x11) \rangle^2}$$

$$= \frac{-3g}{4} \int_0^L dx \left[\left(\frac{2}{L} \sin\left(\frac{\pi x}{L}\right) \right)^2 \right]^2 = \frac{-3g}{4} \frac{4}{L^2} \frac{3L}{8} \boxed{= \frac{9}{8} \frac{g}{L}}$$

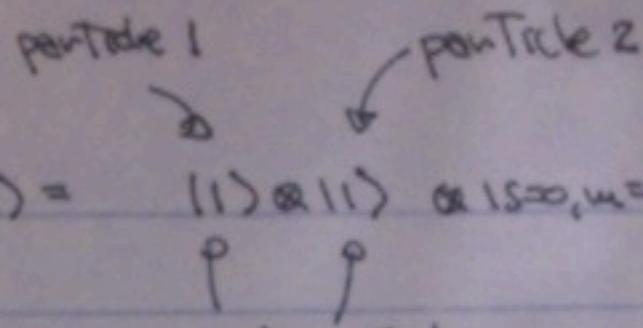
9) For any state w/ total spin 1 (symmetric in spin), the orbital part must be anti-symmetric under the exchange of particles. Thus they have the form $| \text{excited} \rangle = \frac{|1a\rangle \otimes |1b\rangle - |1b\rangle \otimes |1a\rangle}{\sqrt{2}}$ or $\langle s=1, m=1 \rangle$, for some $|1a\rangle$ and $|1b\rangle$. The 1^1S excited state will have $\sqrt{2}$ have $|1a\rangle = |1\rangle$ and $|1b\rangle = |2\rangle$. Computing the perturbation as above we have

$$\langle \text{excited} | -g \hat{\vec{s}}_1 \cdot \hat{\vec{s}}_2 | \text{excited} \rangle = -g \langle s=1, m=1 | \hat{\vec{s}}_1 \cdot \hat{\vec{s}}_2 | s=1, m=1 \rangle$$

$$= \left[\frac{\langle 11 \otimes 21 - 21 \otimes 11 \rangle}{\sqrt{2}} \right] \int dx \langle x|x| \rangle |11 \otimes 21 \rangle$$

$$= \left[\frac{\langle 11 \otimes 12 - 12 \otimes 11 \rangle}{\sqrt{2}} \right]$$

(4)



$$\text{ground state : } |g_{\text{ground}}\rangle = |1\rangle \otimes |1\rangle \text{ or } |s=0, m=0\rangle$$

single particle
ground state

The shift in energy is given by 1st order pert. theory as

$$\Delta E = \langle g_{\text{ground}} | -g \hat{\vec{s}}_1 \cdot \hat{\vec{s}}_2 \int dx \langle x|x| \rangle |g_{\text{ground}}\rangle$$

$$= -g \underbrace{\langle s=0, m=0 | \hat{\vec{s}}_1 \cdot \hat{\vec{s}}_2 | s=0, m=0 \rangle}_{-\frac{3}{4}} \langle 11 \otimes 11 | \int dx \langle x|x| \rangle |11 \otimes 11 \rangle$$

$$= \frac{-3g}{4} \underbrace{\int dx \langle 11x \rangle \langle x11 \rangle}_{\langle (x11) \rangle^2}$$

$$= \frac{-3g}{4} \int_0^L dx \left[\left(\frac{2}{L} \sin\left(\frac{\pi x}{L}\right) \right)^2 \right]^2 = \frac{-3g}{4} \frac{4}{L^2} \frac{3L}{8} \boxed{= \frac{9}{8} \frac{g}{L}}$$

9) For any state w/ total spin 1 (symmetric in spin), the orbital part must be anti-symmetric under the exchange of particles. Thus they have the form $| \text{excited} \rangle = \frac{|1a\rangle \otimes |1b\rangle - |1b\rangle \otimes |1a\rangle}{\sqrt{2}}$ or $\langle s=1, m=1 \rangle$, for some $|1a\rangle$ and $|1b\rangle$. The 1^1S excited state will have $\sqrt{2}$ have $|1a\rangle = |1\rangle$ and $|1b\rangle = |2\rangle$. Computing the perturbation as above we have

$$\langle \text{excited} | -g \hat{\vec{s}}_1 \cdot \hat{\vec{s}}_2 | \text{excited} \rangle = -g \langle s=1, m=1 | \hat{\vec{s}}_1 \cdot \hat{\vec{s}}_2 | s=1, m=1 \rangle$$

$$= \left[\frac{\langle 11 \otimes 21 - 21 \otimes 11 \rangle}{\sqrt{2}} \right] \int dx \langle x|x| \rangle |11 \otimes 21 \rangle$$

$$= \left[\frac{\langle 11 \otimes 12 - 12 \otimes 11 \rangle}{\sqrt{2}} \right]$$

$$= -\frac{g}{4} \frac{1}{q} \int dx \left[C_1(x) C_2(x) C_{11}(x) C_{12}(x) + C_2(x) C_{11}(x) C_{12}(x) C_{11}(x) \right. \\ \left. = C_2(x) C_{11}(x) C_{11}(x) C_{12}(x) - C_{11}(x) C_2(x) C_{12}(x) C_{11}(x) \right]$$

$$= 0$$

This reflects the fact that the perturbation acts only when the two particles are on top of each other. ~~Because~~ But in a state w/ anti-symmetric wavefunctions this never happens.