MIDTERM
Quantum Physics (PHYS 402)

PROBLEM I

Consider a system described by the hamiltonian $\hat{H} = \frac{E_0}{2}(|a\rangle\langle a| - |b\rangle\langle b|) + \sqrt{\frac{3}{4}}(|a\rangle\langle b| + |b\rangle\langle a|)$, with E_0 is real and $\{\ |a\rangle, |b\rangle\ \}$ an orthonormal basis.

1) Show that $|A\rangle = \frac{\sqrt{3}|a\rangle+|b\rangle}{2}$ and $|B\rangle = \frac{|a\rangle-\sqrt{3}|b\rangle}{2}$ are eigenvectors of $\hat{H}$.

2) Find the eigenvalues of $\hat{H}$.

The system is now prepared at the initial instant $t = 0$ in the state $|\psi(t = 0)\rangle = |a\rangle$.

3) Find the state of the system at a later time t.

4) If the energy is measured at a later time t, what are the possible outcomes and with which probabilities ?

5) At time t the energy is measured and the system is found to be in its ground state. What is the probability of finding the system in its ground state at a subsequent time $T > t$?

PROBLEM II

Two identical spin 1/2 fermions are constrained to move in one dimension under the influence of an infinite square well potential of size L.

6) Assuming the particles do not interact among themselves, find out the total spin of the ground state.

7) Compute

$$\begin{aligned} &\langle s = 0, m = 0|\hat{\mathbf{S}}_1.\hat{\mathbf{S}}_2|s = 0, m = 0\rangle, \\ &\langle s = 1, m = 1|\hat{\mathbf{S}}_1.\hat{\mathbf{S}}_2|s = 1, m = 1\rangle, \end{aligned} \tag{1}$$

where $|s = 0, m = 0\rangle, |s = 1, m = 0\rangle$ are the (spin parts of the) states of two spin 1/2 particles with total spin $s = 0, 1$ and z-projections of the total spin equal to $m = 0, 1$ and $\hat{\mathbf{S}}_1, \hat{\mathbf{S}}_2$ are the spin operators of particles 1 and 2.

8) Assume now that there is a zero range spin dependent force force between the fermions described by the interaction

$$\hat{H}_{int} = -g\ \hat{\mathbf{S}}_1.\hat{\mathbf{S}}_2 \int dx\ |x, x\rangle\langle x, x|, \tag{2}$$

where $|x_1, x_2\rangle = |x_1\rangle \otimes |x_2\rangle$ is the eigenstate of the position operators of the two particle $(\hat{x}_1, \hat{x}_2)$ with eigenvalues x_1 and x_2. Compute the shift in energy of the ground state to first order in g.

9) Repeat the item 8) above for the case of the first excited state of the system whose total spin is $s = 1$.

May be useful:

$$\begin{aligned} \int_0^L dx \sin^2\left(\frac{\pi x}{L}\right) &= \frac{L}{2}, \\ \int_0^L dx \sin^4\left(\frac{\pi x}{L}\right) &= \frac{3L}{8}, \\ \int_0^L dx \sin^6\left(\frac{\pi x}{L}\right) &= \frac{5L}{16}, \\ \int_0^L dx \sin^8\left(\frac{\pi x}{L}\right) &= \frac{35L}{128}, \end{aligned} \tag{3}$$