Solution #7

Question A: Perturbation Theory for the Anharmonic oscillator

The anharmonic oscillator is described by the hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 + \lambda\hat{x}^4$. Its eigenvalues cannot be obtained analytically. However for small enough λ we can use perturbation theory taking the harmonic oscillator as the unperturbed hamiltonian. Compute the shift in the energy of the ground state up to order λ^2. Hint: the best way to compute the required matrix elements is to use raising and lowering operators so you don't have to compute any integral. But if you use the explicit form of the wave function you'll end up with integrals of the gaussian type that are not too bad.

Solution:
Basically given this anharmonic oscillator, we can write
$\hat{H} = \hat{H}_0 + \hat{H}', \hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2$, perturbation term $\hat{H}' = \lambda\hat{x}^4$
The zero order for ground state, which is not perturbed, is given as we know:
$\hat{H}_0 |\psi_n^0\rangle = \hat{E}_0 |\psi_n^0\rangle, E_n^0 = (n+\frac{1}{2})\hbar\omega$
For 1st order correction of ground state: $E_0^1 = \langle\psi_0^0|\hat{H}'|\psi_0^0\rangle = \langle\psi_0^0|\lambda\hat{x}^4|\psi_0^0\rangle$.
Using ladder operators, $\hat{x}$ can be written in terms of $\hat{a}_+$and $\hat{a}_-$: $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-)$
Then $E_0^1 = \lambda(\frac{\hbar}{2m\omega})^2 \langle\psi_0^0|(a_+ + a_-)^4|\psi_0^0\rangle = 3\lambda(\frac{\hbar}{2m\omega})^2$
For 2nd order correction of ground state,
$E_0^2 = \sum \frac{|\langle\psi_m^0|\hat{H}'|\psi_0^0\rangle|^2}{E_0^0 - E_m^0} = \sum \frac{|\langle\psi_m^0|\lambda\hat{x}^4|\psi_0^0\rangle|^2}{E_0^0 - E_m^0}, \psi_m^0 = \frac{1}{\sqrt{m!}}(\hat{a}_+)^m\psi_0$
$\langle\psi_m^0|\lambda\hat{x}^4|\psi_0^0\rangle = \langle\psi_m^0|\frac{1}{\sqrt{m!}}\hat{a}_-^m(\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-))^4|\psi_0^0\rangle$
After expansion of $\hat{a}_-^m(\hat{a}_+ + \hat{a}_-))^4$, we find only m=2 or 4 contributes to an equal number of $\hat{a}_+$ and $\hat{a}_-$.
When m=2, $\langle\psi_2^0|\lambda\hat{x}^4|\psi_0^0\rangle = \frac{\lambda\hbar^2}{4m^2\omega^2}\frac{1}{\sqrt{2}}\langle\psi_0^0|\hat{a}^2(3\hat{a}_-^2 + 2\hat{a}_-^2 + \hat{a}_-^2)|\psi_0^0\rangle = \frac{\lambda\hbar^2}{4m^2\omega^2}\frac{1}{\sqrt{2}}(3+2+1) = \frac{3\lambda\hbar^2}{\sqrt{2}m^2\omega^2}$;
When m=4, $\langle\psi_4^0|\lambda\hat{x}^4|\psi_0^0\rangle = \frac{\lambda\hbar^2}{4m^2\omega^2}\frac{1}{\sqrt{4!}}\langle\psi_0^0|\hat{a}_+^4\hat{a}_-^4|\psi_0^0\rangle = \frac{\sqrt{3}\lambda\hbar^2}{\sqrt{2}m^2\omega^2}$
Hence, $E_0^2 = \frac{|\langle\psi_2^0|\lambda\hat{x}^4|\psi_0^0\rangle|^2}{E_0^0 - E_2^0} + \frac{|\langle\psi_4^0|\lambda\hat{x}^4|\psi_0^0\rangle|^2}{E_0^0 - E_4^0} = -\frac{21\lambda^2\hbar^3}{8m^4\omega^5}$