Solution #6

Question A:
Consider two identical spin-12 fermions in an external harmonic oscillator potential. There is no interaction between the fermions. Write down the eigenstates of the energy for the first three lowest energy states. Make sure you include all states with the same energy.

Solution:
For particles in harmonic oscillator potential, the eigenstates are given by:
$E_n = (n + \frac{1}{2}), n = 0, 1, 2...$
$\psi_n(x) = (\frac{m\omega}{\pi\hbar})^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\frac{\xi^2}{2}}, \xi = \sqrt{\frac{m\omega}{\hbar}}$
Composite wave function for fermions: $\psi_-(\vec{r_1}, \vec{r_2}) = \frac{1}{\sqrt{2}}[\psi_a(\vec{r_1})\psi_b(\vec{r_2}) - \psi_b(\vec{r_1})\psi_a(\vec{r_2})]$
Thus, from simple calculus, we know the first three lowest energy states. They are:
Ground state, $E = E_0 + E_1 = 2\hbar\omega, \psi = \frac{1}{\sqrt{2}}[\psi_0(\vec{r_1})\psi_1(\vec{r_2}) - \psi_1(\vec{r_1})\psi_0(\vec{r_2})]$
First excited state, $E = E_0 + E_2 = 3\hbar\omega, \psi = \frac{1}{\sqrt{2}}[\psi_0(\vec{r_1})\psi_1(\vec{r_2}) - \psi_1(\vec{r_1})\psi_0(\vec{r_2})]$
Second excited states1, $E = E_0 + E_3 = 4\hbar\omega, \psi = \frac{1}{\sqrt{2}}[\psi_0(\vec{r_1})\psi_3(\vec{r_2}) - \psi_3(\vec{r_1})\psi_0(\vec{r_2})]$
Second excited states2, $E = E_1 + E_2 = 4\hbar\omega, \psi = \frac{1}{\sqrt{2}}[\psi_1(\vec{r_1})\psi_2(\vec{r_2}) - \psi_2(\vec{r_1})\psi_1(\vec{r_2})]$

Question B:
See Griffths 5.31

Solution:
From Eq. 5.113, $\frac{E}{V} = \int_0^\infty \rho(\omega)d\omega = \frac{\hbar}{\pi^2 c^3}\int_0^\infty \frac{\omega^3}{(e^{\frac{\hbar\omega}{kT}}-1)}d\omega$. $Define\ x = \frac{\hbar\omega}{kT}$, then:
$\frac{E}{V} = \frac{\hbar}{\pi^2 c^3}(\frac{kT}{\hbar})^4 \int_0^\infty \frac{x^3}{e^x - 1}dx = \frac{\hbar}{\pi^2 c^3}(\frac{kT}{\hbar})^4 \Gamma(4)\zeta(4) = \frac{\pi^2 k^4 T^4}{15 c^3 \hbar^3} \simeq 7.566 \times 10^{-16} T^4 (\frac{J}{m^3 K^4})$

Question C:
See Griffths 5.32

Solution:
$|n\rangle = \psi_n(x) = (\frac{m\omega}{\pi\hbar})^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\frac{\xi^2}{2}}, \xi = \sqrt{\frac{m\omega}{\hbar}}$
$< x >_0 = < x >_1 = 0$
$< x^2 >_0 = \int_0^\infty \psi_0^\dagger\ x^2\ \psi_0 dx = \langle 0|\ x^2\ |0\rangle = \frac{\hbar}{2m\omega}$
$< x^2 >_1 = \int_0^\infty \psi_1^\dagger\ x^2\ \psi_1 dx = \langle 1|\ x^2\ |1\rangle = \frac{3\hbar}{2m\omega}$
$< x >_{01} = \langle 0|\ x\ |1\rangle = \sqrt{\frac{2\hbar}{m\omega}}$
(a) From Eq. 5.19 $\Rightarrow < (x_1 - x_2)^2 >_d = \frac{\hbar}{2m\omega} + \frac{3\hbar}{2m\omega} - 0 = \frac{2\hbar}{m\omega}$
(b) From Eq. 5.21 $\Rightarrow < (x_1 - x_2)^2 >_+ = \frac{2\hbar}{m\omega} - \frac{2\hbar}{2m\omega} = \frac{\hbar}{m\omega}$
(c) From Eq. 5.21 $\Rightarrow < (x_1 - x_2)^2 >_- = \frac{2\hbar}{m\omega} + \frac{2\hbar}{2m\omega} = \frac{3\hbar}{m\omega}$