Solutions to HW5
Quantum Physics II, Fall 2012

Oct. 3rd, 2012

Question A:
A spin lees particle moves in one dimension and, at some instant, is described by the wave function $\psi(x) = \langle x| |\psi\rangle$. At that instant the momentum of the particle is measured. What are the possible outcomes of this measurement and with which probabilities (probability densities, to be more precise) ?

Solution:
To measure momentum, we need to express the wave function in eigen basis of momentum. That means to find $\tilde{\psi}(p)$ from the current postion based wave function $\psi(x)$.
$\tilde{\psi}(p) = \langle p| |\psi\rangle = \int dx \langle p| |x\rangle \langle x| |\psi\rangle$
Since $\hat{p} = -i\hbar\frac{d}{dx}$, $\langle p| |x\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{-\frac{ipx}{\hbar}}$, we have $\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}}\int e^{-\frac{ipx}{\hbar}}\psi(x)dx$.
Thus, the possible outcomes of momentum measurement should be in range of $[-\infty, \infty]$, with corresponding probability of $|\tilde{\psi}(p)|^2$

Question B:
See Griffths 4.49

Solution:
(1)Since the wave function is normalized, $1 = |A|^2(1+4+4) = 9|A|^2$
$\Rightarrow A = \frac{1}{3}$
(2)As we know, $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ are two eigenvalues of S_z.
In this case, $P_{\frac{\hbar}{2}} = \frac{5}{9}$; $P_{-\frac{\hbar}{2}} = \frac{4}{9}$, $< S_z >= \frac{5}{9} \times \frac{\hbar}{2} + \frac{4}{9} \times (-\frac{\hbar}{2}) = \frac{\hbar}{18}$
(3) From Eq. 4.151,
for value of $\frac{\hbar}{2}$: $c_+^{(x)} = (\chi_+^{(x)})\ddagger\chi = \frac{1}{3}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1-2i \\ 2\end{pmatrix} = \frac{3-2i}{3\sqrt{2}}$, then $P_{\frac{\hbar}{2}} = |c_+^{(x)}|^2 = \frac{13}{18}$;
for value of $-\frac{\hbar}{2}$: $c_-^{(x)} = (\chi_-^{(x)})\ddagger\chi = \frac{1}{3}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}1-2i \\ 2\end{pmatrix} = -\frac{1+2i}{3\sqrt{2}}$, then $P_{-\frac{\hbar}{2}} = |c_-^{(x)}|^2 = \frac{5}{18}$.
(4) For S_y, $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ are two eigenvalues as well. $\chi_+^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 \\ i\end{pmatrix}$; $\chi_-^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 \\ -i\end{pmatrix}$.
for $\frac{\hbar}{2}$: $c_+^{(y)} = (\chi_+^{(y)})\ddagger\chi = \frac{1}{3}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix}\begin{pmatrix}1-2i \\ 2\end{pmatrix} = \frac{1-4i}{3\sqrt{2}}$, $P_{\frac{\hbar}{2}} = |c_+^{(y)}|^2 = \frac{17}{18}$;
for $-\frac{\hbar}{2}$: $c_-^{(y)} = (\chi_-^{(y)})\ddagger\chi = \frac{1}{3}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\end{pmatrix}\begin{pmatrix}1-2i \\ 2\end{pmatrix} = \frac{1}{3\sqrt{2}}$, $P_{-\frac{\hbar}{2}} = |c_-^{(y)}|^2 = \frac{1}{18}$.

Question C:
See Griffths 5.1

Solution:
(1) $(m_1+m_2)\vec{R} = m_1\vec{r_1} + m_2\vec{r_2} = m_1\vec{r_1} + m_2(\vec{r_1} - \vec{r}) = (m_1+m_2)\vec{r_1} - m_2\vec{r}$
$\Rightarrow \vec{r_1} = \vec{R} + \frac{m_2}{m_1+m_2}\vec{r} = \vec{R} + \frac{\mu}{m_1}\vec{r}$
$(m_1+m_2)\vec{R} = m_1(\vec{r_2}+\vec{r}) + m_2\vec{r_2}$
$\Rightarrow \vec{r_2} = \vec{R} - \frac{m_1}{m_1+m_2}\vec{r} = \vec{R} - \frac{\mu}{m_2}\vec{r}$
Let $\vec{R} = (X, Y, Z), r = (x, y, z)$.
$(\bigtriangledown_1)x = \frac{\alpha}{\alpha x_1} = \frac{\alpha X}{\alpha x_1}\frac{\alpha}{\alpha X} + \frac{\alpha x}{\alpha x_1}\frac{\alpha}{\alpha x} = \frac{m_1}{m_1+m_2}\frac{\alpha}{\alpha X} + 1 \times \frac{\alpha}{\alpha x} = \frac{\mu}{m_2}(\bigtriangledown_R)_x + (\bigtriangledown_r)_x$

$\Rightarrow \bigtriangledown_1 = \frac{\mu}{m_2} \bigtriangledown_R + \bigtriangledown_r$
$(\bigtriangledown_2)x = \frac{\alpha}{\alpha x_2} = \frac{\alpha X}{\alpha x_2}\frac{\alpha}{\alpha X} + \frac{\alpha x}{\alpha x_2}\frac{\alpha}{\alpha x} = \frac{m_2}{m_1+m_2}\frac{\alpha}{\alpha X} - 1 \times \frac{\alpha}{\alpha x} = \frac{\mu}{m_1}(\bigtriangledown_R)_x - (\bigtriangledown_r)_x$
$\Rightarrow \bigtriangledown_2 = \frac{\mu}{m_1} \bigtriangledown_R - \bigtriangledown_r$
$(2)\bigtriangledown_1^2\psi = \bigtriangledown_1 \cdot (\bigtriangledown_1\psi) = \bigtriangledown_1 \cdot [\frac{\mu}{m_2} \bigtriangledown_R \psi + \bigtriangledown_r\psi] = \frac{\mu}{m_2} \bigtriangledown_R \cdot (\frac{\mu}{m_2} \bigtriangledown_R \psi + \bigtriangledown_r\psi) + \bigtriangledown_r \cdot (\frac{\mu}{m_2} \bigtriangledown_R \psi + \bigtriangledown_r\psi)$
$= (\frac{\mu}{m_2})^2 + 2\frac{\mu}{m_2}(\bigtriangledown_r \cdot \bigtriangledown_R\psi) + \bigtriangledown_r^2\psi$
Likewise, $\bigtriangledown_2^2\psi = (\frac{\mu}{m_1})^2 \bigtriangledown_R^2 \psi - 2\frac{\mu}{m_1}(\bigtriangledown_r \cdot \bigtriangledown_R\psi) + \bigtriangledown_r^2\psi$
$\Rightarrow H\psi = -\frac{\hbar^2}{2m_1} \bigtriangledown_1^2 \psi - \frac{\hbar^2}{2m_2} \bigtriangledown_2^2 \psi + V(\vec{r_1}, \vec{r_2})\psi = E\psi$
$Substitute \bigtriangledown_2^2 \psi, \bigtriangledown_2^2\psi, \Rightarrow -\frac{\hbar^2}{2}[\frac{\mu^2}{m_1 m_2}(\frac{1}{m_1} + \frac{1}{m_2})\bigtriangledown_R^2) + (\frac{1}{m_1} + \frac{1}{m_2})\bigtriangledown_r^2)]\psi + V(\vec{r}) = E\psi$
$Substitute \frac{1}{\mu}, \Rightarrow -\frac{\hbar^2}{2(m_1+m_2)} \bigtriangledown_R^2 \psi - \frac{\hbar^2}{2\mu} \bigtriangledown_r^2 \psi + V(\vec{r})\psi = E\psi$
(3) Put in $\psi = \psi_r(\vec{r})\psi_R(\vec{R})$, and divide by $\psi_r\psi_R :$
$[-\frac{\hbar^2}{2(m_1+m_2)}\frac{1}{\psi_R} \bigtriangledown_R^2 \psi_R] + [-\frac{\hbar^2}{2\mu} \bigtriangledown_r^2 \psi_r + V(\vec{r})] = E.$
As we see, the first term depends only on $\vec{R}$, while the second on $\vec{r}$. So either must be a constant. Let's name them $E_R, E_r respectively$. For sure, $E_R + E_r = E$, and
$-\frac{\hbar^2}{2(m_1+m_2)}\frac{1}{\psi_R} \bigtriangledown_R^2 \psi_R = E_R\psi_R;$
$-\frac{\hbar^2}{2\mu} \bigtriangledown_r^2 \psi_r + V(\vec{r})\psi_r = E_r\psi_r$