Solutions to HW4
Quantum Physics II, Fall 2012

September 27, 2012

Q_1:
The z-component of the spin of an electron is measure and the value $\frac{\hbar}{2}$ is found. Immediately afterwards, the spin along a direction making an angle θ with the z-axis is measured. What are the possible outcomes of this second measurement and with which probabilities they arise?

Solution:
As we know, the z-component of spin $S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. Its eigenstate and corresponding eigenvalue is $\frac{\hbar}{2}$: $|\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $-\frac{\hbar}{2}$: $|\downarrow\rangle = \begin{pmatrix}0\\-1\end{pmatrix}$ Since $\frac{\hbar}{2}$ was measured, the state of that moment shoude be $|\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix}$
For the second measurement, it's easier just to use the eigenstates of the operator S_θ which corresponds to the spin measured along the new axis. These eigenstates were derived in class and turn out to be $\frac{\hbar}{2} : |\theta, \uparrow\rangle = \begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix}$; $-\frac{\hbar}{2} : |\theta, \downarrow\rangle = \begin{pmatrix}e^{-i\phi}\sin\frac{\theta}{2}\\ -\cos\frac{\theta}{2}\end{pmatrix}$, ϕ is the angle formed by the projection of new axis and x-axis in XOY plane.

Now we express the state in basis of the eigenstates of the operator S_θ.

$$\begin{pmatrix}1\\0\end{pmatrix} = \cos\frac{\theta}{2}\begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix} + e^{i\phi}\sin\frac{\theta}{2}\begin{pmatrix}e^{-i\phi}\sin\frac{\theta}{2}\\ -\cos\frac{\theta}{2}\end{pmatrix}$$

We see the probability for measuring $P(\frac{\hbar}{2}) = |\cos(\frac{\theta}{2})|^2$, $P(-\frac{\hbar}{2}) = |\sin(\frac{\theta}{2})|^2$
For sure, the total probabilities for this measurement is 1.

Question B(Griffths 4.50):
Suppose two spin-1/2 particles are known to be in the singlet configuration(Equation 4.178). Let $S_a^{(1)}$ be the component of the spin angular momentum of particle a number 1 in the direction defined by the unit vector $\hat{a}$. Similarly, let $S_b^{(2)}$ be the component of the (2)'s angular momentum in the direction $\hat{b}$. Show that
$< S_a^{(1)} S_b^{(2)} > = -\frac{\hbar^2}{4}\cos\theta$, where θ is the angle between $\hat{a}$ and $\hat{b}$.

Solution:
From Equation 4.178, the singlet configuration is $|00\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle)$ Then we choose axes so that $\hat{a}$lies along the z axis and $\hat{b}$ is in the xz plane. $S_a^{(1)} = S_z^{(1)}$, $S_b^{(2)} = \cos\theta S_z^{(2)} + \sin\theta S_x^{(2)}$

$S_a^{(1)} S_b^{(2)} |00\rangle = \frac{1}{\sqrt{2}}[S_z^{(1)}(\cos\theta S_z^{(2)} + \sin\theta S_x^{(2)})](\uparrow\downarrow - \downarrow\uparrow)$
$= \frac{1}{\sqrt{2}}[(S_z \uparrow)(\cos\theta S_z \downarrow) - (S_z \downarrow)(\cos\theta S_z \uparrow + \sin\theta S_x \uparrow)]$
$= \frac{1}{\sqrt{2}}(\frac{\hbar}{2} \uparrow)[\cos\theta(-\frac{\hbar}{2} \downarrow) + \sin\theta(\frac{\hbar}{2} \uparrow)] - (-\frac{\hbar}{2} \downarrow)[\cos\theta(-\frac{\hbar}{2} \uparrow) + \sin\theta(\frac{\hbar}{2} \downarrow)](using Eq. 4.145) = \frac{\hbar^2}{4}[\cos\theta \frac{1}{\sqrt{2}}(- \uparrow\downarrow$
$+ \downarrow\uparrow) + \sin\theta \frac{1}{\sqrt{2}}(\uparrow\uparrow + \downarrow\downarrow)]$
$= \frac{\hbar^2}{4}[-\cos\theta |00\rangle + \sin\theta \frac{1}{\sqrt{2}}(|11\rangle + |1-1\rangle)]$

So, $< S_a^{(1)} S_b^{(2)} >= \langle 00| S_a^{(1)} S_b^{(2)} |00\rangle$
$= \frac{\hbar^2}{4} \langle 00| [-\cos\theta |00\rangle + \sin\theta \frac{1}{\sqrt{2}}(|11\rangle + |1-1\rangle)]$
$= -\frac{\hbar^2}{4} \cos\theta \langle 00| |00\rangle$
$= -\frac{\hbar^2}{4} \cos\theta$