QUANTUM PHYSICS I

PROBLEM SET 1

due September 5, before class

A. I. Exercise your math muscles

1) compute i^i
2) compute $e^{i\pi/2}$
3) find the general solution to

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x), \tag{1}$$

for $E > 0$.
4) what is the solution to the problem above satisfying the conditions

$$\psi(0) = 1, \qquad \frac{d\psi(x)}{dx}|_{x=0} = 0\ ? \tag{2}$$

5) Solve

$$\frac{df(t)}{dt} = Af(t),\ \ f(0) = B. \tag{3}$$

6) Solve

$$\frac{d^2y(x)}{dx^2} = Ay(x),\ \ y(0) = B, \frac{dy}{dx}|_{x=0} = 0, \quad A > 0. \tag{4}$$

6) Solve

$$\frac{d^2y(x)}{dx^2} = Ay(x),\ \ y(0) = B, \frac{dy}{dx}|_{x=0} = 0, \quad A < 0. \tag{5}$$

B. II. All you wanted to know about Dirac's δ-function and were afraid to ask

We can define the δ-function buy its behavior inside integrals:

$$\int_a^b f(x)\delta(x-y) \equiv f(y), \quad \text{for} \quad a < y < b, \tag{6}$$

for any well-behaved function $f(x)$, usually called the *test-function.* You can assume these test-functions are always well behaved and go to zero at infinity. Show that
a) $\int_{-\infty}^{\infty} dx(x^3-1)\delta(x-1) = 0$
b) $\delta(cx) = \frac{1}{|c|}\delta(x)$ (Hint: Insert both sides of the equation in the definition of δ above and change variables.)
c) $\frac{d\theta(x)}{dx} = \delta(x)$ where $\theta(x)$ is the step function

$$\theta(x) = \begin{cases} 1 & \text{if } x > 0; \\ 0 & \text{if } x < 0. \end{cases} \tag{7}$$

(and $\theta(0) = 1/2$ if it ever matters).
d) What is the Fourier transform of $\delta(x)$

$$F(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-ikx}\delta(x) =? \tag{8}$$

Use Plancherel's theorem (see text) to show that

$$\delta(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx}, \tag{9}$$

which is a relation we used in class after a hand waving "proof".

e) Show that

$$\int_{-\infty}^{\infty} dx f(x) \delta'(x) = -f'(0). \tag{10}$$

Feel free to assume that $f(x) \to$ as fast as necessary as $x \to \pm\infty$.

f) Another way of defining the δ-function is through the relation

$$\delta(x) \equiv \lim_{\alpha \to \infty} \sqrt{\frac{\alpha}{\pi}} e^{-\alpha x^2}. \tag{11}$$

Show that the result of d) is the same using this new definition. Feel free to exchange the order of limits and integrations and assume that the test functions are as well behaved as necessary, we are all friends here.