BRA's & KET's

(or linear algebra in Dirac's notation)

- Kets ($|\varphi\rangle$), are vectors, except they live in spaces with dimension $\neq 3$. We could write $\vec{\varphi}$, but we don't, we write

$$|\varphi\rangle$$

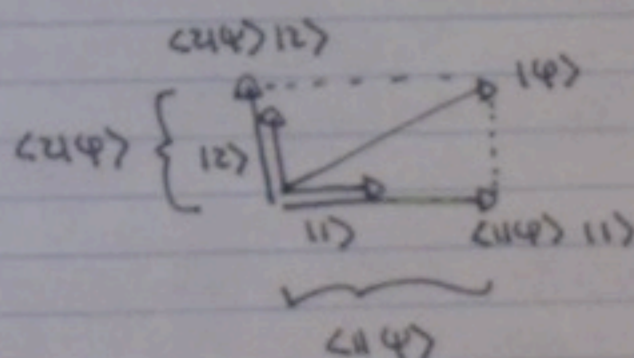
- The other difference is that kets live in a complex linear space, that is, they can be multiplied by complex numbers

$$\alpha|\varphi\rangle = |\varphi\rangle\alpha = \text{ket}$$

complex number

- Just like regular vectors, sometimes we want to do calculations in a specific coordinate system (or basis). We can write a ket in terms of a basis $\{|1\rangle, |2\rangle, \dots\}$ as

$$|\varphi\rangle = \sum_n \underbrace{\langle n|\varphi\rangle}_{\text{scaler product between } |\varphi\rangle \text{ and } |n\rangle} |n\rangle$$



$$= \sum_n \varphi_n |n\rangle$$

coordinates of $|\varphi\rangle$ in the $\{|n\rangle\}$ basis

The scalar product can be a complex number. Also, in a complex space

$$\langle\varphi|\psi\rangle = \langle\psi|\varphi\rangle^*$$

don't forget!

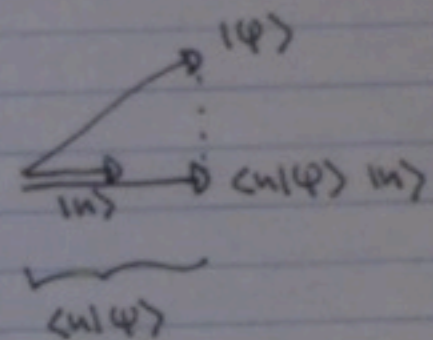
- Frequently, we want to do explicit calculations using a coordinate system (or a basis). Orthonormal basis have the property

orthonormal: $\langle n|m\rangle = \delta_{nm}$

complete: $\underbrace{\sum_n |n\rangle\langle n|}_{} = \mathbb{1}$

projects a ket on the direction $|n\rangle$:

$|n\rangle\langle n|\varphi\rangle = \varphi_n\, |n\rangle$



The same ket can be represented in different basis

different basis

$$|\varphi\rangle = \underbrace{\sum_n \underbrace{\langle n|\varphi\rangle}_{\equiv \varphi_n} |n\rangle}_{} = \underbrace{\sum_n \underbrace{\langle \tilde{n}|\varphi\rangle}_{\equiv \tilde{\varphi}_n} |\tilde{n}\rangle}_{}$$

different coordinates

same ket

- linear operators (take a ket and produce another ket) also have coordinates

$$\hat{A}|\psi\rangle = |\varphi\rangle \Rightarrow \underbrace{\langle n|\varphi\rangle}_{\varphi_n} = \langle n|\hat{A}\,\mathbb{1}|\psi\rangle = \langle n|\hat{A}\sum_m |m\rangle\langle m|\psi\rangle$$

φ_n = nth coordinate of $\hat{A}|\psi\rangle$ in the $\{|n\rangle\}$ basis

$$= \sum_m \underbrace{\langle n|\hat{A}|m\rangle}_{\equiv A_{nm}} \underbrace{\langle m|\psi\rangle}_{\psi_m}$$

- We can give a meaning to $\langle\psi|$ by itself. It's ~~something~~ linear function that takes a ket and produces a number, namely, the scalar product between $|\psi\rangle$ and the given ket:

$$\underbrace{\langle\psi|}_{\text{bra}}\,\underbrace{\big(|\varphi\rangle\big)}_{\text{ket}} = \underbrace{\langle\psi|\varphi\rangle}_{\text{bracket = number}}$$

feed a ket to $\langle\psi|$

get a number

- The above rule relating a ket $|\psi\rangle$ to $\langle\psi|$ is denoted by a dagger and called hermitian conjugation

$$\langle\psi| = |\psi\rangle^{\dagger}$$

note that:

$$\begin{aligned}
(\alpha|\psi\rangle)^{\dagger}|\varphi\rangle &= \big[\langle\varphi|(\alpha|\psi\rangle)\big]^{*} \\
&= \big[\alpha\ \langle\varphi|\psi\rangle\big]^{*} \\
&= \alpha^{*}\ \langle\varphi|\psi\rangle^{*} \\
&= \alpha^{*}\ \langle\psi|\varphi\rangle
\end{aligned}$$

$$\Downarrow$$

$$(\alpha|\psi\rangle)^{\dagger} = \alpha^{*}\ \langle\psi|$$

(d)

Sometimes we organize the components φ_n as a column and A_{nm} as a matrix. Then $|\varphi\rangle = \hat{A}|\psi\rangle$ turns into

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & \cdots \\ A_{21} & A_{22} & \cdots \\ \vdots & & \ddots \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \end{pmatrix}$$

Some of you only knew linear algebra in this matrix language. Now you know better.

By the way, the equation above explain why matrix multiplication is defined in that weird "row-times-column" manner.

- hermitian operators are those satisfying

$$\langle\varphi|\hat{A}|\psi\rangle = \langle\psi|\hat{A}|\varphi\rangle^*$$

Their matrix elements in a orthonormal basis satisfy

$$A_{nm} = \langle n|\hat{A}|m\rangle = \langle m|\hat{A}|n\rangle^* = A^*_{mn}$$

or

$$A_{12} = A^*_{21}\ ,\ A_{13} = A^*_{31}\ ,\ A_{11} = \text{real}\ ,\ A_{22} = \text{real}\ , \ldots$$

- The eigenvectors of a hermitian operator form an orthonormal basis and its eigenvalues are real:

$$\hat{A}|n\rangle = a_n|n\rangle$$

— a_n is real

— $\langle n|m\rangle = \delta_{nm}$ (orthonormal)

— $\sum_n |n\rangle\langle n| = \mathbb{1}$ (complete set)

- For a particle moving on a (continuous) line, the eigenvalues of the position operator $\hat{x}$ can be any real number

$$\underbrace{\hat{x}}_{\text{position operator}} \; \underbrace{|x\rangle}_{\text{position eigenstate corresponding to eigenvalue } x} = \underbrace{x}_{\text{eigenvalue}} \; |x\rangle$$

We can use $\{|x\rangle\}$ as a basis

$$|\psi\rangle = \mathbb{1}\,|\psi\rangle = \underbrace{\int dx}_{\substack{\text{sum turns into}\\ \text{an integral since}\\ \text{the index "x" is continuous}}} \; |x\rangle \underbrace{\langle x|\psi\rangle}_{\psi(x)} = \int dx \; \psi(x) \; |x\rangle$$

we call this the wavefunction

~~What is the wave function corresponding to an eigenstate of position?~~

$$\cancel{\underbrace{\langle \hat{x} | x_0\rangle}_{\text{eigenstate of } \hat{x} \text{ w/ eigenvalue } x_0}}$$

The matrix elements of $\hat{x}$ in the $\{|x\rangle\}$ basis are

$$\langle x|\hat{x}|y\rangle = \langle x|y|y\rangle = y\,\langle x|y\rangle = y \underbrace{\delta(x-y)}_{\substack{\text{not } \delta_{xy} \text{ because}\\ x,y \text{ are continuous}}} = x\,\delta(x-y)$$

Let us write the eigenvalue/eigenvector equation for $\hat{x}$ in the $\{|x\rangle\}$ basis:

$$\sum_m X_{nm}\,\psi_m = x\,\psi_n \Rightarrow \int_{-\infty}^{\infty} dz \; y\,\delta(y-z)\,\psi_x(z) = x\,\psi_x(y)$$

$$\Rightarrow y\,\psi_x(y) = x\,\psi_x(y)$$

$m \to z$

$\sum_m \to \int dz$

$n \to y$

$$\Rightarrow \psi_x(y) = \delta(x-y)$$

scalar product in the $\{|x\rangle\}$ basis :

$$\langle\varphi|\psi\rangle = \langle\varphi|\mathbb{1}|\psi\rangle = \int dx\, \underbrace{\langle\varphi|x\rangle}_{\varphi^*(x)} \underbrace{\langle x|\psi\rangle}_{\psi(x)}$$

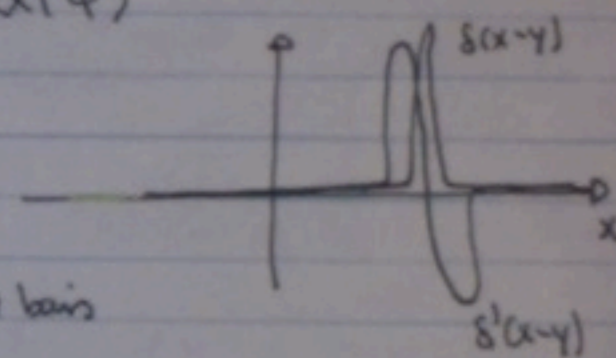
$$= \int dx\, \varphi^*(x)\,\psi(x)$$

notice the similarity to $\varphi_1^*\psi_1 + \varphi_2^*\psi_2 + \dots$

- Another useful basis is formed by eigenfunctions of the momentum operator $\hat{p}$. In the $\{|x\rangle\}$ basis $\hat{p}$ is defined by

$$\langle x|\hat{p}|\psi\rangle = -i\hbar \frac{d}{dx}\psi(x) = -i\hbar\frac{d}{dx}\langle x|\psi\rangle$$

$$\Rightarrow \langle x|\hat{p}|y\rangle = -i\hbar\frac{d}{dx}\delta(x-y)$$



What are the eigenkets of $\hat{p}$? In the $\{|x\rangle\}$ basis they are given by:

$$\hat{p}|p\rangle = p|p\rangle \Rightarrow \langle x|\hat{p}|p\rangle = p\langle x|p\rangle$$

$$\Rightarrow -i\hbar\frac{d}{dx}\langle x|p\rangle = p\langle x|p\rangle$$

$$\Rightarrow \langle x|p\rangle = \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}}$$

normalization chosen so $\langle p|p'\rangle = \delta(p-p')$

coordinates of $|\psi\rangle$ in the $\{|p\rangle\}$ basis:

$$|\psi\rangle = \mathbb{1}|\psi\rangle = \int dp\, |p\rangle\langle p|\psi\rangle = \int dp \underbrace{\int dx \frac{e^{-ipx/\hbar}}{\sqrt{2\pi\hbar}}\psi(x)}_{\langle p|\psi\rangle = \tilde{\psi}(p)} |p\rangle$$

= Fourier Transform of $\psi(x)$

Let us now apply the bra & ket-ology we learned to the measurement problem (postulate 2 of our review of QM). Given the ket $|\psi\rangle$ describing the system at a time of measuring an observable A, the rule giving the probability of finding the value a_n can be summarized as:

eigenvalue problem: $\hat{A}|n\rangle = a_n|n\rangle$ $\Rightarrow$ $\underbrace{|\psi\rangle = \sum_n \langle n|\psi\rangle\, |n\rangle}_{\text{decompose } |\psi\rangle \text{ in the } \{|n\rangle\} \text{ basis}}$

eigenvalue

eigenvector corresponding to eigenvalue a_n

$$\Rightarrow \quad \underbrace{p_n}_{\text{probability of measuring } a_n} = |\langle n|\psi\rangle|^2$$

Let us see how this works when measuring the position of particle moving in 1 dimension. The relevant operator is $\hat{x}$ with eigenvectors $|x\rangle$. We can decompose the ket $|\psi\rangle$ in this basis

$$|\psi\rangle = \int dx \underbrace{\langle x|\psi\rangle}_{\psi(x)} |x\rangle$$

Rule 2 says that the probability (actually, probability <u>density</u>) of finding the particle at some location x_0 is

$$p(x_0) = |\langle x_0|\psi\rangle|^2 = |\psi(x_0)|^2$$

Of course, this is the rule you learned in QM 1.

What if we measured the momentum $\hat{p}$? Then we'd have to expand $|\psi\rangle$ in the $|p\rangle$ basis.

$$|\psi\rangle = \int dp \, \underbrace{\langle p|\psi\rangle}_{\tilde{\psi}(p)} |p\rangle$$

The probability (density) of finding the value p_0 when measuring the momentum is given by

$$P(p_0) = |\langle p_0|\psi\rangle|^2 = |\tilde{\psi}(p_0)|^2,$$

again, as you learned in QM I.