Lecture 7 Highlights

Spin is a purely quantum mechanical quantity that can be represented as a 2-component vector in Hilbert space. Review Griffiths' Appendix on Linear Algebra (page 435), and the discussion of Hilbert space in Chapter 3. We (arbitrarily) choose the basis as follows:

"up" spin $(+) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

"down" spin $(-) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, where "up" and "down" essentially mean $\vec{S}$ is "parallel" or "anti-parallel" to the z-axis (really it means S_z eigenvalue m_s = +1/2 or -1/2).

The operators that manipulate these spinors can be represented as 2x2 matrices that operate on the spin states through matrix-vector multiplication. The S^2 operator has eigenvalue $3\hbar^2/4$ and has a simple form:

$$S^2 = \frac{3\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

When S^2 operates on either (+) or (-) it yields the same vector back, times $3\hbar^2/4$.

On can construct S_z, S_+ and S_- using the reasoning discussed in Griffiths, pages 173, 174. This yields expressions for the 3 components of $\vec{S}$ in terms of the famous Pauli spin matrices:

$$\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where $S_x = \frac{\hbar}{2}\sigma_x, S_y = \frac{\hbar}{2}\sigma_y, S_z = \frac{\hbar}{2}\sigma_z$. The Pauli spin matrices have the interesting property that $\sigma_i^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. This follows from the fact that two consecutive measurements of the same component of S must yield the eigenvector times a positive eigenvalue.

The spin operators S_x, S_y, S_z do not commute (HW2, problem 4.26). They are incompatible operators, and one cannot create simultaneous eigenfunctions of two or more of these three operators. Spin is another example of a ladder of states, in this case a two-rung ladder. All of the general properties we discussed about ladders of states in the context of orbital angular momentum ($\vec{L}, L^2, L_z$, etc.) apply also to spin.

Another important property of the Pauli spin matrices is the fact that they <u>anti-</u>commute (check it using the above matrices!). This is often written as:

$$\sigma_i\sigma_j + \sigma_j\sigma_i = \lfloor \sigma_i, \sigma_j \rfloor_+ = 0$$

Anti-commutation is a common property of Fermionic operators (more on that later).

We also started Griffiths problem 4.27 in class. There we saw the importance of the adjoint, or complex conjugate transpose. This turns a column vector into a complex conjugate row vector, for instance.

The Stern-Gerlach (SG) device uses an inhomogeneous magnetic field to exert a force on neutral atoms in a manner that depends on the sign and magnitude of the "z-component" of magnetic moment (anti-parallel to the spin of the electron) of the atom. The z-direction is defined by the direction of the gradient in magnetic field of the SG device (see Griffiths page 181). The SG takes un-polarized atoms as input and produces spin-polarized beams as output. Atoms in these beams are in S_z eigenstates.