Lecture 5 Highlights

We began by reviewing the ladder operators $a_\pm$ for the one-dimensional harmonic oscillator [Griffiths page 42]. These operators move one up and down the ladder of equally-spaced energy eigenstates (ψ_n) of the 1D harmonic oscillator:

$$a_+\psi_n = \sqrt{n+1}\,\psi_{n+1} \qquad\qquad a_-\psi_n = \sqrt{n}\,\psi_{n-1}$$

Each state has an associated eigen-energy $E_n = (n+1/2)\hbar\omega$. The ladder of states is semi-infinite: it has a lowest rung (the ground state, $n=0$) but there is no upper limit on the value of n.

One can define similar operators for the z-component of angular momentum, L_z, and they are:

$$L_\pm = L_x \pm iL_y$$

These operators move one up and down a ladder of states of fixed L^2 eigenvalue, and the rungs are labeled by the L_z component eigenvalue $m\hbar$. The proof follows Griffiths, pages 162-166. The 'ladder of states' is finite in this case because the z-component of $\vec{L}$ can never exceed the total length of $\vec{L}$ (i.e. the square root of the L^2 eigenvalue). One finds that the ladder of states is symmetric about a z-component value of zero, and that there are an integer number of states (N) between the bottom rung and the top rung of the ladder. If the maximum eigenvalue of L_z on the ladder is called $\ell\hbar$, then the minimum value is $-\ell\hbar$, and the L^2 operator has eigenvalue $\ell(\ell+1)\hbar^2$. Note that since $\ell < \sqrt{\ell(\ell+1)}$ for non-zero ℓ, the $\vec{L}$ vector never points precisely in the z-direction. The most important result is that $\ell = N/2$, meaning that ℓ is either an integer or half-integer (or zero). For the orbital angular momentum of the electron in the hydrogen atom ℓ is an integer or zero. Soon we will consider other types of angular momentum in which the largest z-component eigenvalue is a half-integer.