**Lecture 39 Highlights**

We are now going to use the WKB approximation to calculate tunneling rates through odd-shaped barriers. You encountered tunneling before in Phys 401, including:

1) Section 2.5 where tunneling through a delta-function barrier of height $V(x) = \alpha\ \delta(x)$. The result for the transmission coefficient is $T = \frac{1}{1 + E_0 / E}$, where $E$ is the energy of the particle and $E_0 = m\alpha^2 / 2\hbar^2$ is a characteristic energy in the problem.

2) Problem 2.33 Tunneling through a rectangular barrier of height $V_0$ and width $a$. The transmission coefficient is $T = \frac{1}{1 + \frac{V_0^2}{4E(V_0 - E)}\sinh^2\left(\frac{a}{\hbar}\sqrt{2m(V_0 - E)}\right)}$ valid for $E < V_0$.

Consider the WKB approximation for particles in the "classically forbidden" region, $E < V(x)$. In this region the kinetic energy $p^2 / 2m$ is negative, which can be interpreted as resulting from an imaginary momentum $p = \sqrt{2m(E - V(x))}$. In this case the WKB solutions we found last time become:

$$\psi(x) = \frac{D}{\sqrt{|p_{class}(x)|}} \exp\left[\pm \frac{1}{\hbar}\int^x |p_{class}(x')| dx'\right],$$

with $p_{class} = \sqrt{2m(E - V(x))}$. Notice that the complex exponential has now become a positive or negative exponential because the solutions are no longer running waves.

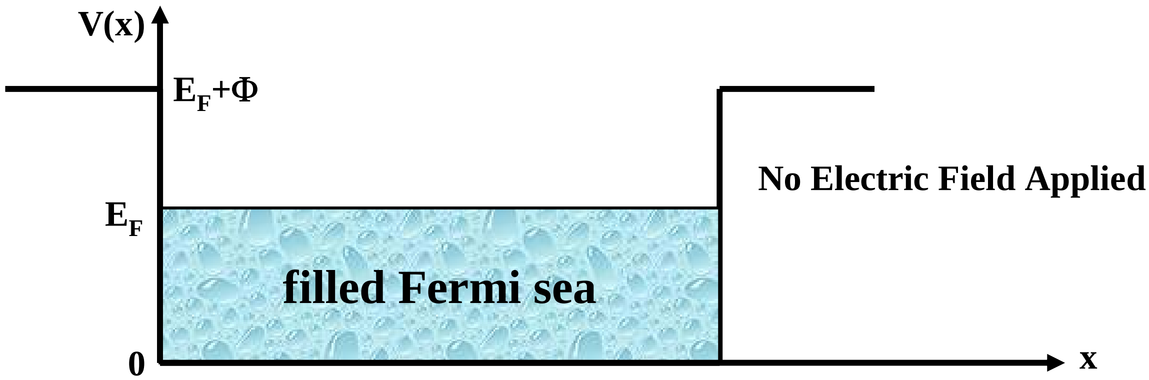
Consider a barrier of width $a$ with an arbitrary potential on top, $V(x)$, as discussed on pages 320-322 of Griffiths. For energies $E$ less than the minimum of $V(x)$, and for barriers that are sufficiently tall and thick, the transmission probability is dominated by the negative exponential term;

$$T \propto e^{-2\gamma}, \text{ where } \gamma = \frac{1}{\hbar}\int_0^a |p_{class}(x')| dx'$$

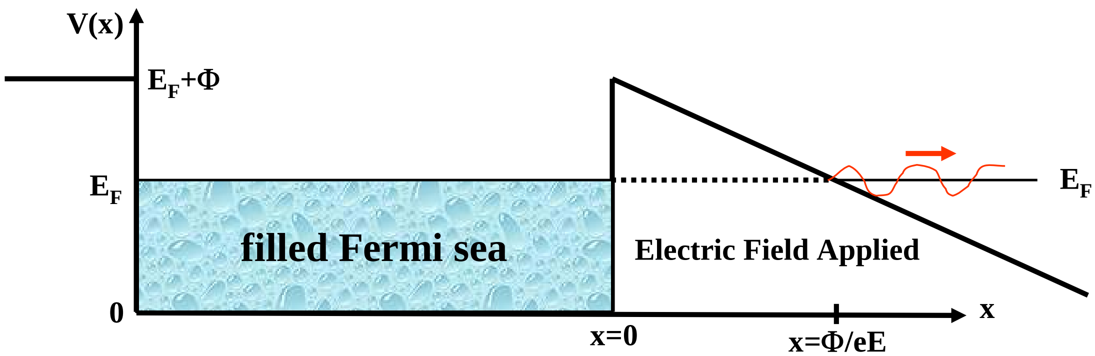
Going back to the flat-top barrier of problem 2.33 to test this result, the transmission probability in the WKB approximation (after doing the integral with $V(x) = V_0$) is $T \propto e^{-\frac{2a}{\hbar}\sqrt{2m(V_0 - E)}}$. If we expand the exact result given above in the limit of tall and wide barrier (i.e. $\frac{a}{\hbar}\sqrt{2m(V_0 - E)} >> 1$), the result is

$T \cong \frac{4E(V_0 - E)}{V_0^2} e^{-\frac{2a}{\hbar}\sqrt{2m(V_0 - E)}}$. The pre-factor is on the order of 1, so the exponential dominates, and is in agreement with the WKB approximate result.

Now consider the problem of cold emission from a metal. A metal can be modeled as a potential well of depth $E_F + \Phi$, where $E_F$ is the Fermi energy and $\Phi$ is the work function of the metal (see the Figure below). The work function is the energy required to remove an electron from the Fermi energy in the metal and to set it free. Work functions are typically on the order of a few electron volts for metals.




If an electric field is applied to the surface of a metal, the potential is modified and it becomes possible for electrons to tunnel out of the metal into free space (this is called cold emission by Fowler-Nordheim tunneling).




Note that the tunnel barrier is not a "flat top" but instead triangular shaped. The potential is given by $V(x) = E_F + \Phi - eEx$, where $E$ now (confusingly!) is the applied electric field. The calculation of the Fowler-Nordheim tunneling rate is a job for the WKB approximation!

$$T \propto e^{-2\gamma}, \text{ where } \gamma = \frac{1}{\hbar}\int_0^a |p_{class}(x')|dx',$$

and $\gamma = \frac{1}{\hbar}\int_0^{\Phi/eE}\sqrt{2m(\Phi - eEx')}dx'$. The upper limit of the integral is the point in $x$ where the Fermi energy is equal to the potential outside the metal (a classical turning point). The integral can be solved by standard methods and the result is;

$$T = \exp\left[-\frac{4}{3}\frac{\sqrt{2m}}{\hbar}\frac{\Phi^{3/2}}{eE}\right]$$

The tunneling current is proportional to the transmission probability. Hence the log of the tunnel current should be proportional to the inverse of the electric field strength. This is the characteristic of Fowler-Nordheim tunneling, and data showing this behavior is posted on the class web site.