Lecture 30 Highlights

First we will calculate the number of ways that n_s particles can be distributed into state s of energy E_s and degeneracy g_s. This will be called P_s. Next we will calculate the total number of arrangements for an entire set of occupation numbers $n_1, n_2, n_3, n_4, \ldots n_s, \ldots$ This will be the statistical weight W of the arrangement ($n_1, n_2, n_3, n_4, \ldots n_s, \ldots$):

$$W(n_1, n_2, ..., n_s, ...) = \prod_{s=1}^{\infty} P_s$$

This weight will be proportional to the probability of finding this particular distribution of occupation numbers.

The next step is to maximize W by varying all of the occupation number values subject to the number and total energy constraints. We will then do thermodynamics with the most probable microscopic configuration.

Consider 3 cases:

1) Distinguishable classical particles
2) Indistinguishable identical Fermions
3) Indistinguishable identical Bosons

Distinguishable classical particles: This is something of a fiction in the sense that each particle has a unique identity and we can keep track of its location and energy with arbitrary precision. Start with the ground state (i = 1, energy E_1 with degeneracy g_1). How many ways are there to put n_1 distinguishable particles in this energy level? The answer is;

$P_1 = \binom{N}{n_1} g_1^{n_1}$, where the binomial coefficient is $\binom{N}{n_1} = \dfrac{N!}{n_1!(N-n_1)!}$. The binomial coefficient arises because we can distinguish each particle and there are many distinct ways to choose a subset n_1 of all the particles N, without regard to order. The particles can each be put into any of g_1 possible states, hence the factor of $g_1^{n_1}$.

When constructing P_2 there is a similar factor, except that there are now only $N-n_1$ particles to start with. Hence $P_2 = \binom{N-n_1}{n_2} g_2^{n_2}$, and so on. When we construct the relative statistical weight W, the result has a lot of cancellation:

$$W_{Dist}(n_1, n_2, ..., n_s, ...) = \frac{N!\, g_1^{n_1}}{n_1!(N-n_1)!} \times \frac{(N-n_1)!\, g_2^{n_2}}{n_2!(N-n_1-n_2)!} \times \frac{(N-n_1-n_2)!\, g_3^{n_3}}{n_3!(N-n_1-n_2-n_3)!} \times ...$$

$$W_{Dist}(n_1, n_2, ..., n_s, ...) = N! \prod_{s=1}^{\infty} \frac{g_s^{n_s}}{n_s!}$$

Indistinguishable identical Fermions: In this case we do not have the problem of choosing n_1 particles out of N since they are all completely identical and there is no need to enumerate how such choices can be made – there is only one way. Instead we are now concerned with enforcing the Pauli exclusion principle. In this case it means that n_s must be less than or equal to g_s, but never greater. If n_s is less than g_s we have the freedom to distribute the particles many different ways. In fact there are $\begin{pmatrix} g_s \\ n_s \end{pmatrix}$ ways to put the n_s particles into the g_s available states. Note that if $n_s = g_s$, this reduces to a factor of 1 since there is only one way to distribute one of the identical particles to each available quantum state. Similarly if $n_s = 0$ there is only one way to accomplish that, so $P_s = 1$. Now the statistical weight is;

$$W_{Fermions}(n_1, n_2, ..., n_s, ...) = \prod_{s=1}^{\infty} \frac{g_s!}{n_s!(g_s - n_s)!}$$

Indistinguishable identical Bosons: From the treatment in Griffiths, one finds the result for the statistical weight is:

$$W_{Bosons}(n_1, n_2, ..., n_s, ...) = \prod_{s=1}^{\infty} \frac{(n_s + g_s - 1)!}{n_s!(g_s - 1)!}$$

The next step is to maximize $W(n_1, n_2, ..., n_s, ...)$ by varying all of the occupation numbers, subject to the number and total energy constraints: $\sum_{i=1}^{\infty} n_i = N$ and $\sum_{i=1}^{\infty} n_i E_i = E$. We will include the constraints using the method of Lagrange multipliers. This method allows one to perform a constrained maximization. We will form a new function to maximize, namely;

$$G(n_1, n_2, ..., n_s, ..., \alpha, \beta) = W(n_1, n_2, ..., n_s, ...) + \alpha\left(N - \sum_{i=1}^{\infty} n_i\right) + \beta\left(E - \sum_{i=1}^{\infty} n_i E_i\right)$$

To maximize this function we must enforce these conditions:

$$\frac{\partial G}{\partial n_s} = 0 \ \forall s \ \text{ and } \ \frac{\partial G}{\partial \alpha} = \frac{\partial G}{\partial \beta} = 0 .$$

The form of *G* already satisfies the last two conditions.