Lecture 21 Highlights

Up to this point we have only considered static solutions to the Schrödinger equation. It is now time to consider what happens to a quantum system when it is given a time-dependent perturbation. The philosophy of this calculation is as follows. Consider a quantum system governed by a time-independent 'baseline' or unperturbed Hamiltonian H^0 that has solutions to the time-dependent Schrödinger equation $H^0\Psi_n^0(\vec{r},t) = i\hbar\frac{d}{dt}\Psi_n^0(\vec{r},t)$ of the form $\Psi_n^0(\vec{r},t) = \phi_n^0(\vec{r})e^{-iE_n^0 t/\hbar}$, where E_n^0 is the unperturbed eigen-energy. Suppose that this system is prepared in a particular eigenstate, say the nth state. Next consider turning on a "small" time-dependent perturbing potential such that the new Hamiltonian is given by $H^0 + \lambda H'(\vec{r},t)$, where $\lambda << 1$ and the perturbation is in general a function of both position and time. Let this perturbation act for some time 't', and then have it stop. Now the system is governed once again by the unperturbed time-independent Hamiltonian H^0. The question is this: what is the probability that the quantum system is now in some other state "j"? This is equivalent to asking for the probability that the system has made a quantum jump from state 'n' to state 'j'.

To address this question we employ a time-dependent version of perturbation theory. While the perturbation is on, the wavefunction becomes $\Psi(\vec{r},t)$ and satisfies the new time-dependent Schrödinger equation:

$$[H^0 + \lambda H'(\vec{r},t)]\Psi(\vec{r},t) = i\hbar\frac{d}{dt}\Psi(\vec{r},t)$$

We employ the trick of expanding the new wavefunction around the unperturbed solution plus a series of ever smaller corrections, $\Psi_n = \Psi_n^0 + \lambda\Psi_n^1 + \lambda^2\Psi_n^2 + \ldots$, and substitute this into the time-dependent Schrödinger equation. Collecting like-powers of λ yields

$\lambda^0 : H^0\Psi_n^0 = i\hbar\frac{d}{dt}\Psi_n^0$, which is the original unperturbed problem,

$\lambda^1 : H^0\Psi_n^1 + H'\Psi_n^0 = i\hbar\frac{d}{dt}\Psi_n^1$. We use the completeness postulate of quantum mechanics to express the first order correction to the wavefunction as an infinite sum over all the unperturbed eigenfunctions: $\Psi_n^1 = \sum_\ell a_{nl}(t)\Psi_\ell^0(\vec{r},t)$ with unknown time-dependent coefficients $a_{nl}(t)$. Substituting this into the λ^1 equation and projecting out the jth eigenstate yields the amplitude transition rate from state 'n' to state 'j':

$$\dot{a}_{nj} = \frac{-i}{\hbar}e^{i(E_j^0 - E_n^0)t/\hbar}\int \phi_j^{0*}(\vec{x})\,\mathrm{H}'(\vec{x},t)\,\phi_n^0(\vec{x})d^3x \qquad (1)$$

Hence if we know the perturbing Hamiltonian, this matrix element can be computed and the result integrated over time to find the transition amplitude from state 'n' to state 'j', $a_{nj}(t)$. The probability of the transition is proportional to $|a_{nj}(t)|^2$.

We then considered two-level systems, as discussed by Griffiths in the first few pages of Chapter 9.

A 2-level system with states ‘a’ and ‘b’ subject to a time-dependent perturbation will have a wavefunction of the form:

$$\Psi(t) = c_a(t)\psi_a e^{-iE_a t/\hbar} + c_b(t)\psi_b e^{-iE_b t/\hbar}$$

Assuming that the system started in state “a” at time t=0, just before the time-dependent perturbation began, gives the initial conditions:

$c_a(0) = 1$, $c_b(0) = 0$.

Demanding that $\Psi(t)$ satisfies the time-dependent Schrödinger equation we can solve for the rate at which amplitude builds up in state ‘b’:

$$\dot{c}_b(t) = \frac{-i}{\hbar} e^{i(E_b - E_a)t/\hbar} \int \psi_b^*(\vec{x})\, H'(\vec{x},t)\, \psi_a(\vec{x})\, d^3x$$

This result is a special case of Eq. (1) above.