**Lecture 2 Summary**
**Phys 402**

We wrote down the Schrödinger equation in spherical coordinates and proceeded to solve it by separation of variables. We will solve it for a Hydrogen atom in which there is a (conservative) electrostatic (central) force between a proton and an electron, with potential $V(r) = -e^2/(4\pi\varepsilon_0 r)$, where $e$ is the electronic charge, $\varepsilon_0$ is the permittivity of free space, and $r$ is the radial coordinate, representing the distance from the proton to the electron (we shall discuss the QM 2-body problem in more detail during a discussion section later in the semester). We look for a solution with constant energy $E$ such that $\psi(r,\theta,\phi,t) = u(r,\theta,\phi)e^{-iEt/\hbar}$. The resulting time-independent Schrödinger equation in spherical coordinates is given by Eq. [4.14] of Griffiths.

Separate variables as $u(r,\theta,\phi) = R(r)Y(\theta,\phi)$ to get an equation that has $r$-dependent terms (only) on one side, and $\theta,\phi$-dependent terms (only) on the other side (Griffiths, p. 134). Each side of the equation must separately equal a constant "$\alpha$" (i.e. something independent of $r,\theta,\phi$), yielding the radial and angular equations, Griffiths [4.16] and [4.17], respectively.

Starting with the definition of the angular momentum operator $\vec{L} = \vec{r}\times\vec{p} = \vec{r}\times(-i\hbar\vec{\nabla})$, the angular momentum squared operator in spherical coordinates is: $L^2 = \vec{L}\bullet\vec{L} = \frac{-\hbar^2}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{\hbar^2}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$. By comparison with Eq. [4.17] one sees that it contains the angular momentum squared operator: $L^2 Y = \hbar^2\alpha Y$, which is a nice eigenvalue problem. The eigenvalues of $L^2$ will turn out to be $\hbar^2\ell(\ell+1)$, where $\ell$ is zero or a positive integer, and the eigenfunction is the 'spherical harmonic' $Y_\ell^m(\theta,\phi)$, where $m$ is another positive or negative integer or zero with $\ell \ge |m|$.

The radial equation has an infinite number of bound states (E<0) for any given value of $\ell$.

$$\frac{-\hbar^2}{2m}\frac{d^2(rR)}{dr^2} + \left[\frac{-e^2}{4\pi\varepsilon_0 r} + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right](rR) = E(rR)$$

The solutions are proportional to a finite polynomial called the Laguerre polynomial, which we will find later. The solution to this equation also results in a quantization condition for the energy: $E_n = -\frac{1}{n^2}\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2$, where $m$ is the electron (reduced) mass, and $n$ is an integer that is bigger than $\ell$, i.e. $\ell \le n-1$. One also finds a characteristic length for the Hydrogen atom, called the Bohr radius: $a_0 = \frac{4\pi\varepsilon_0\hbar^2}{me^2}$, which is about 0.5 Angstroms.

The full solution of the time-independent Schrödinger equation for the H-atom is found by multiplying the R(r) solution with the angular solution and properly normalizing the entire wavefunction:

$$\psi_{n\ell m}(r,\theta,\phi,t) = \sqrt{\left(\frac{2}{na_0}\right)^3 \frac{(n-\ell-1)!}{2n[(n+\ell)!]^3}} \left(\frac{2r}{na_0}\right)^\ell e^{-r/na_0} L_{n-\ell-1}^{2\ell+1}\left(\frac{2r}{na_0}\right) Y_\ell^m(\theta,\phi) e^{-iE_n t/\hbar}$$

There are three quantum numbers: $n$ (principal), $\ell$ (ang. mom.) and $m$ (magnetic). They have possible values given by:

$$n = 1,2,3,4,...$$

$$\ell = 0,1,2,...n-1$$

$$m = -\ell, -\ell+1, ...0, ...\ell-1, \ell$$

The Hydrogen atom wavefunctions are orthonormal, Griffiths [4.90].

We shall examine the angular and radial equations in more detail next.