Lecture 13 Highlights

The eigenfunctions of J^2 can be expressed as linear combinations of states with different values of m_ℓ and m_s using the world-famous Clebsch-Gordan coefficients ($C^{\ell\ s\ j}_{m_\ell\ m_s\ m_j}$) as:

$$\left| j\ m_j \right\rangle = \sum_{m_\ell + m_s = m_j} C^{\ell\ \ s\ \ j}_{m_\ell\ m_s\ m_j} \left| \ell\ m_\ell \right\rangle \left| s\ m_s \right\rangle$$

Where the ket $\left| \ell\ m_\ell \right\rangle$ represents the spherical harmonics $Y_\ell^{m_\ell}$. The C-G coefficient values are given in Table 4.8 on page 188 of Griffiths. Remember that the all of the coefficients should appear under a square root, with the minus sign (if any) out front. Also note that we have dropped the radial part of the wavefunction ($R_{n\ell}$) because it plays no role in combining angular momenta. Don't forget to put it back later.

Where do these coefficients come from? Consider starting with a product wavefunction at the top of the m_j ladder (it is a product of the wavefunctions with maximum values of m_ℓ and m_s). Now apply the J_- lowering operator, and construct orthonormal states on lower rungs of the ladder. The coefficients on the terms of those wavefunctions are the C-G coefficients.

We did a specific example of a hydrogen atom with $\ell = 1$ and spin $s = 1/2$. In this case the angular momentum vector and spin vector can either be "parallel" or "anti-parallel." Consider the two cases:

1) "Parallel" $\vec{L}$ and $\vec{S}$: The maximum value of m_ℓ is +1, while the value of m_s is +1/2 for the "parallel" case. This means that $m_j = m_\ell + m_s = 3/2$. This is the state at the top of the ladder. There must also be states with $m_j = +1/2, -1/2, -3/2$. This is a set of 4 states on the ladder of $j = 3/2$. Thus the eigenvalues of J^2 for this ladder must be $\frac{3}{2}(\frac{3}{2}+1)\hbar^2 = \frac{15}{4}\hbar^2$. Note that $\vec{L} \bullet \vec{S} > 0$ is this case, giving a positive spin-orbit Hamiltonian perturbation.

2) "Anti-Parallel" $\vec{L}$ and $\vec{S}$: The maximum value of m_ℓ is +1, while the value of m_s is -1/2 for the "anti-parallel" case. This means that $m_j = m_\ell + m_s = 1/2$. This is the state at the top of the ladder. There must also be a state with $m_j = -1/2$. This is a set of 2 states on the ladder of $j = 1/2$. Thus the eigenvalues of J^2 for this ladder must be $\frac{1}{2}(\frac{1}{2}+1)\hbar^2 = \frac{3}{4}\hbar^2$. Note that $\vec{L} \bullet \vec{S} < 0$ is this case, giving a negative spin-orbit Hamiltonian perturbation.

There are a total of 6 states possible by simply combining the orbital angular momentum with $\ell = 1$ and spin angular momentum with $s = 1/2$! Just imagine what happens when you combine 3 or more angular momentum vectors!

Now for an example of how to construct states that are simultaneous eigenfunctions of L^2, S^2, J^2 and J_z. Take the case again of hydrogen with $\ell = 1$ and spin $s = 1/2$. How do we find the state with $j = 3/2$ and $m_j = -1/2$ in terms of the $Y_\ell^{m_\ell}$ and spinors? Look at the $1 \times \frac{1}{2}$ Table on page 188. We are led to this table because we are combining an angular momentum vector with $\ell = 1$ and spin vector with $s = 1/2$. Now look under the column labeled " $\begin{matrix} 3/2 \\ -1/2 \end{matrix}$ ". It says:

$$\left|\frac{3}{2} \;\; -\frac{1}{2}\right\rangle = \sum_{m_\ell + m_s = -1/2} C^{1 \;\; 1/2 \;\; 3/2}_{m_\ell \;\; m_s \;\; -1/2} \left|1 \; m_\ell\right\rangle \left|\frac{1}{2} \; m_s\right\rangle$$

$$\left|\frac{3}{2} \;\; -\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}} \left|1 \; 0\right\rangle \left|\frac{1}{2} \;\; -\frac{1}{2}\right\rangle + \sqrt{\frac{1}{3}} \left|1 \;\; -1\right\rangle \left|\frac{1}{2} \;\; \frac{1}{2}\right\rangle$$

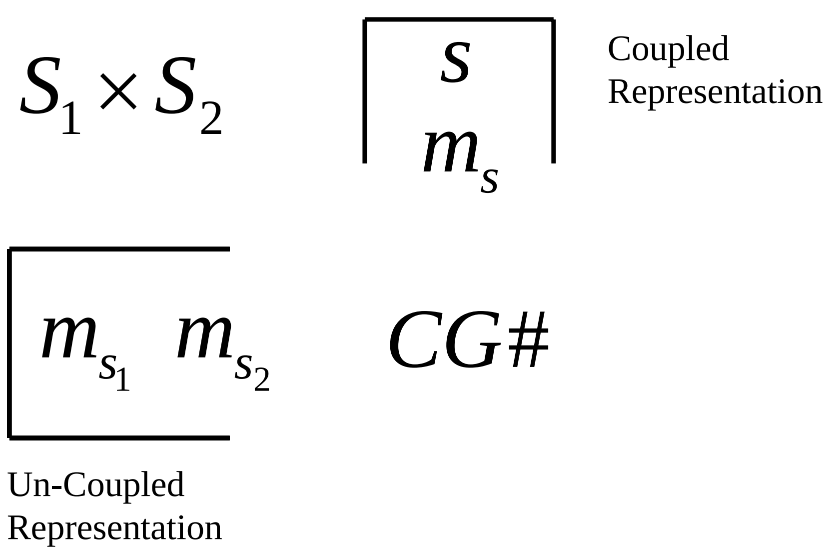
This can be written in a more familiar way in terms of spherical harmonics and spinors as:

$$\left|\frac{3}{2} \;\; -\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}} Y_1^0 \chi_- + \sqrt{\frac{1}{3}} Y_1^{-1} \chi_+$$

One can move back and forth between the coupled and un-coupled representations using the Clebsch-Gordan table on page 188. Here is the schematic layout for the CG table for combining two spins (called $\vec{S}_1$, $\vec{S}_2$) to form a total spin $\vec{S} = \vec{S}_1 + \vec{S}_2$ (S^2 has eigenvalue $s(s+1)\hbar^2$):

General Schematic of the C-G Table

$$\vec{S} = \vec{S}_1 + \vec{S}_2$$



We considered what happens when two spin-1/2 spins are combined. The lecture followed Griffiths pages 184-188, although not in that order. We considered the 4 naïve product states of the two spins:

$$\left|\uparrow\right\rangle\left|\uparrow\right\rangle, \quad \left|\uparrow\right\rangle\left|\downarrow\right\rangle, \quad \left|\downarrow\right\rangle\left|\uparrow\right\rangle, \quad \left|\downarrow\right\rangle\left|\downarrow\right\rangle$$

where $\left|\uparrow\right\rangle\left|\uparrow\right\rangle$ represents the product ket $\left|\frac{1}{2} \quad +\frac{1}{2}\right\rangle_1 \left|\frac{1}{2} \quad +\frac{1}{2}\right\rangle_2$, where the first number in each ket represents s_1 and s_2, respectively, and the second number represents m_{s1} and m_{s2}. We found that the eigen-kets of the S^2 operator (where $\vec{S} = \vec{S}_1 + \vec{S}_2$) are in two ladders of states:

$$\left|1 \quad 1\right\rangle = \left|\uparrow\right\rangle\left|\uparrow\right\rangle$$

$$\left|1 \quad 0\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\downarrow\right\rangle\left|\uparrow\right\rangle + \left|\uparrow\right\rangle\left|\downarrow\right\rangle\right)$$ This is the s = 1 ladder of 3 states. **TRIPLET**

$$\left|1 \quad -1\right\rangle = \left|\downarrow\right\rangle\left|\downarrow\right\rangle$$

and

$$\left|0 \quad 0\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\downarrow\right\rangle\left|\uparrow\right\rangle - \left|\uparrow\right\rangle\left|\downarrow\right\rangle\right)$$ This is the s = 0 ladder of 1 state. **SINGLET**

The kets on the left are written in the “coupled representation” while those on the right are in the “un-coupled representation.” These are 4 orthonormal states that span the Hilbert space for the two spins.

Note that we started with spins that individually live on half-integer ladders, but the combined spin lives on integer ladders. This will have important ramifications for the physics of multi-particle systems, such as multi-electron atoms and the theory of superconductivity, among other things.