Lecture 11 Highlights

We considered the spin-orbit interaction in Hydrogen. The objective is to calculate the "fine-structure" energy splitting due to the spin-orbit effect. The magnetic moment of the electron interacts with the magnetic field created by the proton to produce a small energy difference between the $\vec{L}$ parallel to $\vec{S}$ and $\vec{L}$ anti-parallel to $\vec{S}$ situations for the atom. This difference is due to "spin-orbit coupling."

As discussed in Griffiths page 271 the magnetic field experienced by the electron is given (classically) by:

$$\vec{\mathrm{B}} = \frac{e\vec{L}}{8\pi\varepsilon_0 mc^2 r^3},$$

where e is the electronic charge, m is the electron mass, c is the speed of light in vacuum, and r is the proton-electron distance. Note that the magnetic field is parallel to the electron orbital angular momentum vector. To see why, consider things from the proton's rest frame as the electron moves with velocity $\vec{v}$ through the static electric field $\vec{E}$ produced by the proton. It will experience an effective magnetic field given by $\vec{B} = -\frac{\vec{v}}{c} \times \vec{E}$ (from relativity). By comparing the direction of this field with the direction of the angular momentum of the electron in its orbit about the proton, this argument shows that $\vec{B}$ is parallel to $\vec{L}$.

The magnetic moment of charged "spinning" particles is given by:

$$\vec{\mu} = \gamma \vec{S},$$

where γ is called the gyromagnetic ratio. It relates the gyration (or rotation) of the particle (as embodied in $\vec{S}$) to the magnetic moment developed ($\vec{\mu}$). A moving charge creates a magnetic field. A charge moving in a "small" current loop can be treated as a magnetic moment, or magnetic dipole, at least for distances large compared to the diameter of the current loop. For the electron the gyromagnetic ratio is found to be $\gamma_e = -\frac{e}{m_e}$ to very good approximation. See Griffiths page 272 for a "derivation" of this result. For heavier particles like the proton the gyromagnetic ratio is much smaller due to the larger mass in the denominator and the fact that angular momentum is quantized and of order $\hbar$ for all particles. This fact allows us to ignore the interaction of the proton's magnetic moment with the magnetic field created by the electron, at least for now.

The electron's magnetic moment experiences a torque due to its motion around the proton. There is a perturbing interaction energy given by;

$$\mathrm{H}_{so} = -\vec{\mu} \bullet \vec{B}$$

which becomes;

$$\mathrm{H}_{so} = -\left(-\frac{e}{m}\vec{S}\right) \bullet \left(\frac{e\vec{L}}{8\pi\varepsilon_0 mc^2 r^3}\right) \sim \vec{S} \bullet \vec{L}$$

This new operator $\vec{S} \bullet \vec{L}$ has some interesting properties. It commutes with L^2 and S^2, but does not commute with $\vec{S}$ or $\vec{L}$ (Homework 4). This means that $\vec{S}$ and $\vec{L}$ are no longer "constants of the motion" under the perturbed Hamiltonian $H^0 + H_{so}$ (this follows from Griffiths [3.71] with $Q = \vec{L}$ or $\vec{S}$). This means that ℓ and s are still "good quantum numbers", but m_ℓ and m_s are not. The perturbation mixes together states with different values of m_ℓ and m_s.

Note that the proton exerts a torque on the electron spin. This means that the force of interaction between the two particles is non-central, although this effect is a "small perturbation." This means that $\vec{S}$ will precess in its motion about the proton. As a consequence $\vec{L}$ will also precess, since the net external torque on the atom is zero, and the total angular momentum of the atom, $\vec{J} = \vec{L} + \vec{S}$, must remain fixed.

This new total angular momentum operator $\vec{J}$ has properties analogous to $\vec{S}$ and $\vec{L}$. It has a ladder of states symmetric about zero. The ladder has a top rung and a bottom rung. There is a J^2 operator with eigenvalues $j(j+1)\hbar^2$, and a J_z operator with eigenvalues $m_j\hbar$. There are raising and lowering operators $J_\pm = J_x \pm iJ_y$, and commutators such as $\lfloor J_x, J_y \rfloor = i\hbar J_z$.

One nice feature of $\vec{J}$ is the fact that it is a "constant of the motion" for the perturbed Hamiltonian $H^0 + H_{so}$. Hence although we loose m_ℓ and m_s as good quantum numbers, we gain j and m_j.