Department of Physics
University of Maryland

Physics 273 Fall 2003
Homework Assignment # 1

Due Sep. 8
1.1, 1.2, 1.3(1.6), 2.1, 2.2

Problem Solutions

1.1

In Fig 1.1 (a), the restoring force in the pendulum is

$$F = m\ddot{x} = -mg\sin(\theta) \cong -mg\frac{x}{L}$$
$$m\ddot{x} + mg\frac{x}{L} = 0$$
$$\ddot{x} + \frac{g}{L}x = 0$$

therefore, $\omega^2 = \frac{g}{L}$ and the stiffness, $s = \frac{F}{x} = \frac{mg}{L}$

Check the dimension! $\omega^2 = [\frac{LT^{-2}}{L}] = [T^{-2}]$

For a small angle θ , $\frac{x}{L} = \sin\theta \cong \theta$

$$x = L\theta \quad and \quad \ddot{x} = L\ddot{\theta}$$

$$mL\ddot{\theta} + mg\theta = 0 \qquad \ddot{\theta} + \frac{g}{L}\theta = 0$$

so, we have the same result. $\underline{\omega^2 = \frac{g}{L}}$

In Fig 1.1 (b), mass is replaced by the moment of inertia, I and the stiffness replaced by the restoring couple of the wire, C rad^{-1}.

$$m \to I \quad and \quad s \to C$$

$$m\ddot{x} + s\,x = 0 \quad \Leftrightarrow \quad I\ddot{\theta} + C\,\theta = 0$$
$$\ddot{\theta} + \frac{C}{I}\theta = 0 \qquad \underline{\omega^2 = \frac{C}{I}}$$

Check the dimension! $\omega^2 = [\frac{C}{I}] = [\frac{M\,L\,T^{-2}2\pi L}{M\,L^2}] = [T^{-2}]$

In Fig 1.1 (d), ignoring the force by the gravity, consider tensions of wires.

$$m\ddot{x} \;=\; -\;(T\,\sin(\theta)\;-\;\sin(-\theta))\;=\;-\;2\,T\,\sin(\theta)$$

For a small angle, $\sin(\theta)\;\simeq\;\theta\;\simeq\;\tan(\theta)\;\simeq\;\frac{x}{L}$

$$m\ddot{x} + 2\,T\,\frac{x}{L}\;=\;0$$

$$\ddot{x} + \frac{2\,T}{m\,L}x\;=\;0 \qquad\qquad \underline{\omega^2\;=\;\frac{2\,T}{m\,L}}$$

Check the dimension! $\omega^2\;=\;[\frac{2\,T}{m\,L}]\;=\;[\frac{M\,L\,T^{-2}}{M\,L}]\;=\;[T^{-2}]$

In Fig 1.1 (e), the restoring force is difference of mass in the cylinder,

$$F\;=\;-\,m_x\;g\;=\;-\,\rho\,A\,x\,g$$

So, the stiffness, $s\;=\;\frac{F}{x}\;=\;2\,A\,\rho\,g$

$$F\;=\;m\,\ddot{x}\;=\;-\,\rho\,A\,x\,g$$

$$A\,L\,\rho\,\ddot{x}\;+\;2\rho\,g\,A\,x\;=\;0$$

where L is the total length of the fluid

$$\ddot{x}\;+\;\frac{2g}{L}\,x\;=\;0 \qquad\qquad \underline{\omega^2\;=\;\frac{2g}{L}}$$

Check the dimension! $\omega^2\;=\;[\frac{2\,g}{L}]\;=\;[\frac{L\,T^{-2}}{L}]\;=\;[T^{-2}]$

In Fig 1.1 (f),

$$P\,V^{\gamma}\;=\;C\;=\;constant$$

$$dP\,V^{\gamma}\;+\;\gamma\,P\,V^{\gamma-1}\,dV\;=\;0 \qquad\qquad \frac{dP}{P}\;+\;\gamma\frac{dV}{V}\;=\;0$$

$$dP\;=\;-\,\gamma P\frac{dV}{V}\;=\;\underline{-\,\gamma P\frac{Ax}{V}}$$

$$F\;=\;m\ddot{x}\;=\;dP\cdot A\;=\;-\,\gamma P\frac{Ax}{V}\,A$$

$$\rho\,A\,L\ddot{x} + \gamma P\frac{A}{V}\,x\,A\;=\;0$$

$$\ddot{x} + \frac{\gamma\,P\,A}{L\,\rho\,V}x\;=\;0 \qquad\qquad \underline{\omega^2\;=\;\frac{\gamma\,P\,A}{L\,\rho\,V}}$$

Check the dimension! $\omega^2\;=\;[\frac{\gamma\,P\,A}{L\,\rho\,V}]\;=\;[\frac{1\cdot M\,L\,T^{-2}}{M\,L}]\;=\;[T^{-2}]$

In Fig 1.1 (g), the restoring force is due to the mass of the volume displaced in the neck $F = -\rho A\, x\, g$

$$F = m\,\ddot{x} = -\rho A\, x\, g$$

$$A\,\rho\, g\,\ddot{x} + \rho A\, g\, x$$

$$\ddot{x} + \frac{\rho A\, g}{m} x \qquad \underline{\omega^2 = \frac{\rho A\, g}{m}}$$

Check the dimension! $\omega^2 = [\frac{A\,\rho\, g}{m}] = [\frac{L^2 \cdot M\, L^{-3} M\, L\, T^{-2}}{M}] = [T^{-2}]$

1.2 From the equation (1.2) $x = A\cos(\omega t) + B\sin(\omega t)$

we can use these relations,

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$
$$\sin(A - B) = \sin A\cos B - \cos A\sin B$$
$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$
$$\cos(A - B) = \cos A\cos B + \sin A\sin B$$

(a) When $\underline{A = a\cos(\phi)}$ and $\underline{B = -a\sin(\phi)}$

$$x = A\cos(\omega t) + B\sin(\omega t) = a\cos(\phi)\,\cos(\omega t) - a\sin(\phi)\,\sin(\omega t)$$
$$= a\cos(\omega t + \phi)_\diamond$$

and verify $\ddot{x} + \omega^2 x = 0$

$$\ddot{x} = -\omega^2\, a\cos(\omega t + \phi)$$
$$\ddot{x} + \omega^2 x = -\omega^2\, a\cos(\omega t + \phi) + \omega^2 a\cos(\omega t + \phi) = 0_\diamond$$

(b) When $\underline{A = -a\sin(\phi)}$ and $\underline{B = a\cos(\phi)}$

$$x = A\cos(\omega t) + B\sin(\omega t) = -a\sin(\phi)\,\cos(\omega t) + a\cos(\phi)\,\sin(\omega t)$$
$$= a\,[\sin(\omega t)\,\cos(\phi) - \cos(\omega t)\,\sin(\phi)\,] = a\sin(\omega t - \phi)_\diamond$$

and verify $\ddot{x} + \omega^2 x = 0$

$$\ddot{x} = -\omega^2\, a\sin(\omega t - \phi)$$
$$\ddot{x} + \omega^2 x = -\omega^2\, a\sin(\omega t - \phi) + \omega^2 a\sin(\omega t - \phi) = 0_\diamond$$

(c) When $\underline{A = a\cos(\phi)}$ and $\underline{B = a\sin(\phi)}$

$$x = A\cos(\omega t) + B\sin(\omega t) = a\cos(\phi)\cos(\omega t) + a\sin(\phi)\sin(\omega t)$$

$$= a\cos(\omega t - \phi)$$

and verify $\ddot{x} + \omega^2 x = 0$

$$\ddot{x} = -\omega^2 a\cos(\omega t - \phi)$$

$$\ddot{x} + \omega^2 x = -\omega^2 a\cos(\omega t - \phi) + \omega^2 a\cos(\omega t - \phi) = 0_\diamond$$

1.3 (a) At $t = 0,\ x = a$

1) $x = a\sin(\omega t + \phi) \quad \rightarrow \quad a = a\sin(\phi) \quad \sin(\phi) = 1$

$\underline{\phi = \frac{\pi}{2} + 2n\pi}$ *where* $n = 0, \pm 1, \pm 2 \ldots$

2) $x = a\cos(\omega t + \phi) \quad \rightarrow \quad a = a\cos(\phi) \quad \cos(\phi) = 1$

$\underline{\phi = 2n\pi}$ *where* $n = 0, \pm 1, \pm 2 \ldots$

3) $x = a\sin(\omega t - \phi) \quad \rightarrow \quad a = a\sin(-\phi) \quad -\sin(\phi) = 1$

$\underline{\phi = \frac{3\pi}{2} + 2n\pi}$ *where* $n = 0, \pm 1, \pm 2 \ldots$

4) $x = a\cos(\omega t - \phi) \quad \rightarrow \quad a = a\cos(-\phi) \quad \cos(\phi) = 1$

$\underline{\phi = 2n\pi}$ *where* $n = 0, \pm 1, \pm 2 \ldots$

(b) At $t = 0,\ x = -a$

1) $x = a\sin(\omega t + \phi) \quad \rightarrow \quad -a = a\sin(\phi) \quad \sin(\phi) = -1$

$\underline{\phi = \frac{3\pi}{2} + 2n\pi}$ *where* $n = 0, \pm 1, \pm 2 \ldots$

2) $x = a\cos(\omega t + \phi) \quad \rightarrow \quad -a = a\cos(\phi) \quad \cos(\phi) = -1$

$\underline{\phi = \pi + 2n\pi}$ *where* $n = 0, \pm 1, \pm 2 \ldots$

3) $x = a\sin(\omega t - \phi) \quad \rightarrow \quad -a = a\sin(-\phi) \quad -\sin(\phi) = -1$

$\underline{\phi = \frac{\pi}{2} + 2n\pi}$ *where* $n = 0, \pm 1, \pm 2 \ldots$

4) $x = a\cos(\omega t - \phi) \quad \rightarrow \quad a = a\cos(-\phi) \quad \cos(\phi) = -1$

$\underline{\phi = \pi + 2n\pi}$ *where* $n = 0, \pm 1, \pm 2 \ldots$

For each ϕ, find values of ωt at (c) $x = \frac{a}{\sqrt{2}}$, (d) $x = \frac{a}{2}$, and (e) $x = 0$

From the result of (a),

$$1)\ x = a\sin\left(\omega t + \tfrac{\pi}{2}\right) = a\cos(\omega t)$$

$$2)\quad x = a\cos(\omega t + 0) = a\cos(\omega t)$$

$$3)\quad x = a\sin(\omega t - \tfrac{3\pi}{2}) = a\cos(\omega t)$$

$$4)\quad x = a\cos(\omega t - 0) = a\cos(\omega t)$$

so, these four have same ωt at each position,

(c) $x = \frac{a}{\sqrt{2}} \quad \rightarrow \quad \frac{a}{\sqrt{2}} = a\cos(\omega t) \qquad \underline{\omega t = \pm\frac{\pi}{4} + 2n\pi} \quad n = 0, \pm 1, \pm 2, \ldots$

(d) $x = \frac{a}{2} \quad \rightarrow \quad \frac{a}{2} = a\cos(\omega t) \qquad \underline{\omega t = \pm\frac{\pi}{3} + 2n\pi} \quad n = 0, \pm 1, \pm 2, \ldots$

(e) $x = 0 \quad \rightarrow \quad 0 = a\cos(\omega t) \qquad \underline{\omega t = \pm\frac{\pi}{2} + n\pi} \quad n = 0, \pm 1, \pm 2, \ldots$

Similarly, from the result of (b),

$$1)\quad x = a\sin(\omega t + \tfrac{3\pi}{2}) = -\,a\cos(\omega t)$$

$$2)\quad x = a\cos(\omega t + \pi) = -\,a\cos(\omega t)$$

$$3)\quad x = a\sin(\omega t - \tfrac{\pi}{2}) = -\,a\cos(\omega t)$$

$$4)\quad x = a\cos(\omega t - \pi) = -\,a\cos(\omega t)$$

so, these four have same ωt at each position,

(c) $x = \frac{a}{\sqrt{2}} \quad \rightarrow \quad \frac{a}{\sqrt{2}} = -\,a\cos(\omega t) \qquad \underline{\omega t = \pi \pm \frac{\pi}{4} + 2n\pi} \quad n = 0, \pm 1, \pm 2, \ldots$

(d) $x = \frac{a}{2} \quad \rightarrow \quad \frac{a}{2} = -\,a\cos(\omega t) \qquad \underline{\omega t = \pi \pm \frac{\pi}{3} + 2n\pi} \quad n = 0, \pm 1, \pm 2, \ldots$

(e) $x = 0 \quad \rightarrow \quad 0 = -\,a\cos(\omega t) \qquad \underline{\omega t = \pm\frac{\pi}{2} + n\pi} \quad n = 0, \pm 1, \pm 2, \ldots$

1.6 (273H) $\qquad x = a\sin(\omega t + \phi)$

At time $t = 0$, from a position x_o with a velocity $\dot{x} = v_o$

Show that $\tan(\phi) = \frac{\omega x_o}{v_o}$ and $a = \sqrt{x_o^2 + (\frac{v_o}{\omega})^2}$

$$x = a\sin(\omega t + \phi), \qquad \dot{x} = a\omega\cos(\omega t + \phi)$$

$$x(0) = x_o = a\sin(\phi) \quad \rightarrow \quad \sin(\phi) = \frac{x_o}{a}$$

$$\dot{x}(0) = v_o = a\omega\cos(\phi) \quad \rightarrow \quad \cos(\phi) = \frac{v_o}{a\omega}$$

$$\underline{\tan(\phi) = \frac{x_o/a}{v_o/a\omega} = \frac{x_o\omega}{v_o}}$$

$$1 = \sin^2(\phi) + \cos^2\phi = (\tfrac{x_o}{a})^2 + (\tfrac{v_o}{a\omega})^2$$

$$\underline{a = \sqrt{(\tfrac{x_o}{a})^2 + (\tfrac{v_o}{a\omega})^2}}$$

2.1

$$x = e^{-p\,t}(F\cosh q\,t + G\sinh q\,t)$$

From the initial conditions, at t = 0 , x(0) = F and $\dot{x} = 0$

$$\dot{x} = -p\,e^{-pt}(F\cosh qt + G\sinh qt) \;+\; e^{-pt}(F\sinh qt + G\cosh qt)$$

$$\dot{x}(0) \;=\; -\,p\,F + q\,G = 0$$

so,

$$\frac{G}{F} \;=\; \frac{p}{q} \;=\; \frac{\frac{r}{2m}}{\sqrt{\frac{r^2}{4m^2-\frac{s}{m}}}} \;=\; \frac{1}{\sqrt{1-\frac{4sm}{r^2}}} \;=\; \underline{\frac{r}{\sqrt{r^2-4ms}}}$$

2.2

$$x \;=\; (A+Bt)\,e^{-\frac{rt}{2m}}$$

$$\dot{x} \;=\; B\,e^{-\frac{rt}{2m}} \;+\; (A+Bt)(-\frac{r}{2m})\,e^{-\frac{rt}{2m}}$$

$$\ddot{x} \;=\; -\frac{Br}{2m}\,e^{-\frac{rt}{2m}} \;+\; B(-\frac{r}{2m})\,e^{-\frac{rt}{2m}} \;+\; (A+Bt)(\frac{r}{2m})^2\,e^{-\frac{rt}{2m}}$$

$$\;=\; -\frac{Br}{m}\,e^{-\frac{rt}{2m}}\,e^{-\frac{rt}{2m}} \;+\; (A+Bt)(\frac{r}{2m})^2\,e^{-\frac{rt}{2m}}$$

Then,

$$m\ddot{x} + r\dot{x} + sx \;=\; 0$$

$$m\,[-\frac{Br}{m}\,e^{-\frac{rt}{2m}} \;+\; (A+Bt)\frac{r^2}{4m^2}\,e^{-\frac{rt}{2m}}] \;+\; r\,[B\,e^{-\frac{rt}{2m}} \;+\; (A+Bt)(-\frac{r}{2m})\,e^{-\frac{rt}{2m}}] \;+\; s\,(A+Bt)\,e^{-\frac{rt}{2m}} \;=\; 0$$

$$-Bre^{-\frac{rt}{2m}} \;+\; (A+Bt)\frac{r^2}{4m}\,e^{-\frac{rt}{2m}} \;+\; Br\,e^{-\frac{rt}{2m}} \;+\; (A+Bt)(-\frac{r^2}{2m})\,e^{-\frac{rt}{2m}} \;+\; s\,(A+Bt)\,e^{-\frac{rt}{2m}} \;=\; 0$$

$$(A+Bt)(-\frac{r^2}{4m} \;+\; s)\,e^{-\frac{rt}{2m}} \;=\; 0$$

When $\frac{r^2}{4m^2} \;=\; \frac{s}{m}$,
$x \;=\; (A+Bt)\,e^{-\frac{rt}{2m}}$ satisfies the equation, $m\ddot{x} + r\dot{x} + sx \;=\; 0$