Physics 273 - Homework #6

1) Let's consider a continuous string with the triangular initial shape. Remember, the string is mounted between two fixed wall at x = 0 and x = L, and its equation of motion is the wave equation:

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2}$$

and the general solution is

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)e^{i\omega_n t}, \text{ where } \omega_n = \sqrt{\frac{T}{\rho}}\frac{n\pi}{L},$$

and the $\{C_n\}$ are some set of coefficients which are determined by the initial conditions. In class we will calculate (on Tuesday March 12) the real and imaginary parts of the $\{C_n\}$ for the case where the string has a triangular shape with height = (h) at t = 0:

$$y(x,t=0) = \begin{cases} \frac{2hx}{L} & 0 \le x \le L/2 \\ \frac{2h(L-x)}{L} & L/2 \le x \le L \end{cases}$$

and zero initial velocity:

$$\dot{y}(x,t=0) = 0.$$

The result is

$$\mathrm{Re}(C_n) \equiv a_n = \frac{8h}{n^2\pi^2}(-1)^{(n-1)/2} \text{ for odd (n) and } a_n = 0 \text{ for even (n), and}$$

$$\mathrm{Im}(C_n) \equiv b_n = 0 \text{ for all (n).}$$

Please turn in a plot for each of the following questions:

a) First consider the solution at t = 0. Let the initial height of the triangle be h = 0.5 meters, and the length of the string be 10 meters. Use a computer to draw the solution, but only including the first non-zero term of the infinite sum (just the first normal mode).

b) Continuing to look at the t = 0 solution, draw the solution including just the first two non-zero terms in the infinite sum.

c) Continue as in parts (a) and (b), but now including the first three non-zero terms (Optional: if it's not too much trouble, keep the first 100 non-zero terms).

d) Now keep the first three (or 100) non-zero terms, as in part (c), but this time we will allow the solution to evolve in time. Let the tension in the string be 10 N and the mass density be 0.1 kg/meter. Draw the shape of the rope at t = 0.1 seconds, keeping just the first three (or 100) terms in the sum.

e) Continue as in part (e), but now draw the shape at t = 0.2 seconds.

f) Continue as in parts (e) and (f), but now draw the shape at t = 0.3 seconds.

2) **Ortho-normality of Sine functions.** The Kronecker Delta (δ_{nm}) is defined to be equal to 0 for $n \neq m$, and equal to 1 for n = m. Given this definition, show that

$$\frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \delta_{nm}\,.$$

where (n) and (m) are integers. Explicitly evaluating the integral for

a) the n = m case.

b) the n $\neq$ m case.

Hint: You may use this trigonometric identity: $\sin(u)\sin(v) = \frac{1}{2}\left[\cos(u-v) - \cos(u+v)\right]$.

3) The classical wave equation and its general solutions are given in problem #1. Show that the general solution is correct by explicitly substituting it into the equation of motion.

4) Consider the set of functions $\{e^{in\pi x/L}\}$, where (n) is any positive or negative integer. Show that these functions are orthogonal to each other over the interval (-L, L) by evaluating this integral:

$$\int_{-L}^{L}\left(e^{in\pi x/L}\right)\left(e^{-im\pi x/L}\right)dx\,.$$

Evaluate the integral for the case where n = m and the case where n $\neq$ m.