Homework #10 - Phys 273

1) A perfect harmonic wave has a single frequency and wavelength, and it extends to +/- infinity in both space and time. For example, at t = 0, a perfect harmonic wave would look like:

$$y_{perfect}(x,t=0) = Ce^{ik_0 x}, -\infty < x < \infty$$

where k_0 is the one-and-only wavenumber for this perfect wave.

It's not possible to send a message using this perfect wave, because sending a message means that the wave must be altered in some way. But any alteration of the wave immediately makes the wave imperfect. For example, suppose I wanted to tell someone in New York that my height is (L) meters. One way I could send this information is to create a truncated harmonic wave whose length is (L) at t = 0:

$$y_{signal}(x) = \begin{cases} Ce^{ik_0 x}, \dfrac{-L}{2} < x < \dfrac{L}{2} \\ 0, otherwise \end{cases}$$

The signal appears to be perfect, because it appears to contain only one wavenumber (just like the perfect wave shown above). But because it is truncated, this wave actually contains a range of wavenumbers, and therefore it is not perfect.

a) Show that $y_{signal}(x)$contains a range of wavenumbers by calculating its Fourier Transform, A(k). Hint: A(k) is a completely real function (no imaginary component). Show this explicitly by eliminating all of the (i)'s from your answer.

b) Suppose that L = 1.8 meters (which is my height). Let $k_0 = 100.0\ m^{-1}$, and C = 0.01 meters. Plot A(k) for (k) between 80 m^{-1} and 120 m^{-1}.

c) No calculation needed for this question: make a sketch of what the Fourier Transform of a perfect harmonic wave would look like as a function of (k). (Just think about it and sketch).

2) The dispersion relation for an electromagnetic wave travelling in an ionized gas is

$$\omega(k) = \sqrt{\omega_e^2 + (ck)^2}$$

where (ω_e) is a constant called the electron plasma frequency and (c) is the speed of light in vacuum.

a) Calculate the expression for the phase velocity of this medium as a function of (ω_e), (c), and (k).

b) Show that the ratio of the phase velocity to the speed of light in vacuum is greater than one for all values of (k). In other words, the phase velocity in this medium is larger than the speed of light in vacuum(!).

c) Make a plot of the phase velocity versus (k), for the case where c = 1 and ω_e = 1. Plot from k = 0 to k = 5.

d) Calculate the expression for the group velocity of this medium as a function of (ω_e), (c), and (k).

e) Make a plot of group velocity as a function of (k) for the case where c = 1 and $\omega_e = 1$. Plot from k = 0 to k = 5.

f) Show that the ratio of the group velocity to (c) is less than one for all values of (k). In other words, the group velocity is always less than the speed of light in vacuum.

g) Special relativity says that no information can be transmitted faster than the speed of light in vacuum. Does this medium violate special relativity?