Fourier's Trick relies upon the following mathematical identity:

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \begin{cases} 0, & m \neq n \\ \frac{L}{2}, & m = n \end{cases}$$

We like to write this more compactly:

Define $\boxed{\delta_{mn} \equiv \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}}$ "Kronecker Delta"

Thus $\delta_{11} = 1$, $\delta_{12} = \emptyset$, $\delta_{13} = \emptyset$, $\delta_{22} = 1$, ect

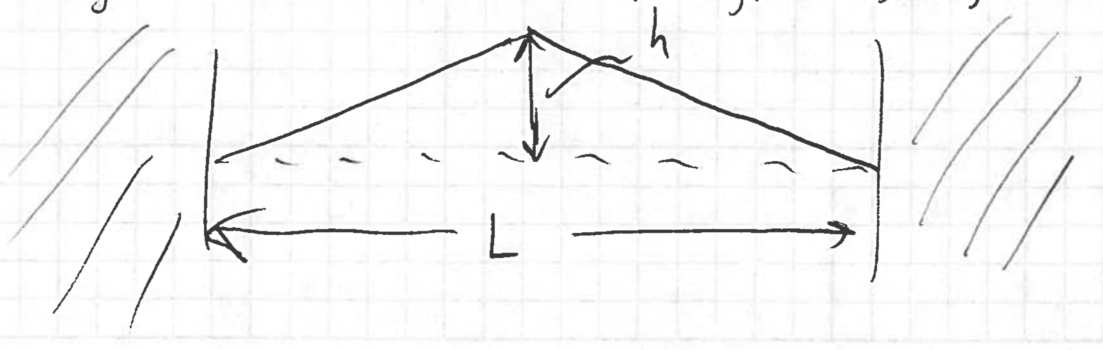
Using the Kronecker Delta we can say

$$\boxed{\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{L}{2}\delta_{mn}}$$

or

$$\boxed{\frac{2}{L}\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \delta_{mn}}$$

Let's go back to our triangular string:



This is the shape at $t = \emptyset$. The functional form is

$$y(x, t=\emptyset) = \begin{cases} \left(\frac{2h}{L}\right) x & , \quad 0 \le x \le \frac{L}{2} \\ \left(\frac{2h}{L}\right)(L-x) & , \quad \frac{L}{2} \le x \le L \end{cases}$$

We want to describe this simple function in a much more complicated way: as an infinite sum of normal modes:

$$y(x, t=\emptyset) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

The question is: what are the $\{a_n\}$?

Fourier's Trick tells us that any particular coefficient, ~~can be calculated~~ for example, the m^{th} coefficient (a_m), can be calculated by evaluating this integral:

$$\boxed{a_m = \frac{2}{L} \int_0^L \sin\left(\frac{m\pi x}{L}\right) y(x)\, dx}$$

For our function $y(x)$, this integral is

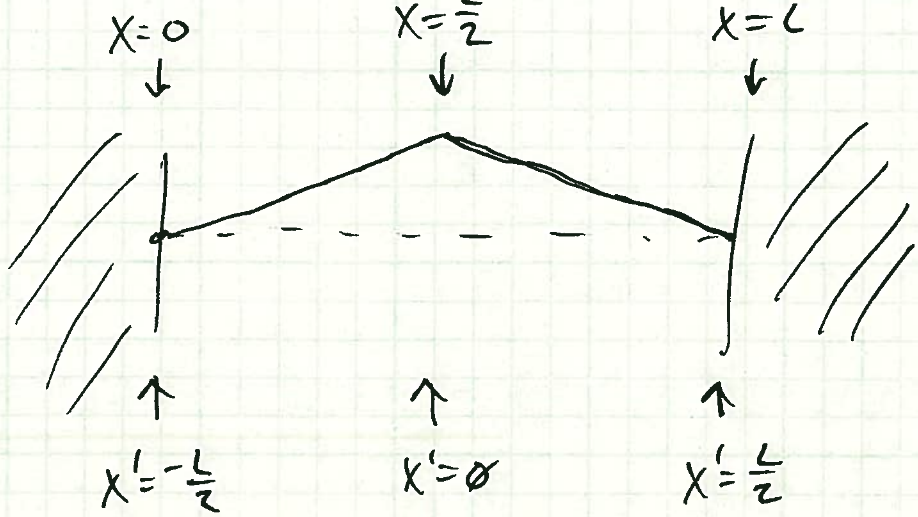
$$a_m = \frac{2}{L} \int_0^{L/2} \sin\left(\frac{m\pi x}{L}\right)\left(\frac{2hx}{L}\right) dx + \frac{2}{L}\int_{\frac{L}{2}}^{L} \sin\left(\frac{m\pi x}{L}\right)\left(\frac{2h(L-x)}{L}\right) dx$$

It turns out that the easiest way to evaluate this integral is to move our coordinate system-

$$\text{Let} \quad x' \equiv x - \frac{L}{2}$$

$$\text{so that} \quad x = x' + \frac{L}{2}$$

This means that $x' = \emptyset$ is the center of the string



In terms of x', our string position at $t = \emptyset$ is

$$\boxed{y(x', t=\emptyset) = \begin{cases} \left(\frac{2h}{L}\right)\left(x' + \frac{L}{2}\right), & -\frac{L}{2} \le x' \le \emptyset \\ \left(\frac{2h}{L}\right)\left(-x' + \frac{L}{2}\right), & 0 \le x' \le \frac{L}{2} \end{cases}}$$

Note that y <u>is an even function of</u> x'.

Also, we have the following math theorem:

If $x = x' + \frac{L}{2}$,

Then $\sin\left(\frac{m\pi x}{L}\right) = \begin{cases} (-1)^{(m-1)/2} \cos\left(\frac{m\pi x'}{L}\right), & \text{for } m = \text{odd} \\ (-1)^{m/2} \sin\left(\frac{m\pi x'}{L}\right), & \text{for } m = \text{even} \end{cases}$

Now our integral has 2 cases:

$$a_m = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} y(x')\,(-1)^{(m-1)/2} \cos\left(\frac{m\pi x'}{L}\right)dx' \quad \text{for } m=\text{odd}$$

AND

$$a_m = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \underbrace{y(x')\,(-1)^{(m-1)/2} \sin\left(\frac{m\pi x'}{L}\right)}\,dx' \quad \text{for } m=\text{even}$$

This integrand is an ~~even~~ <u>odd</u> function of x', because $y(x')$ is even, and $\sin\left(\frac{m\pi x'}{L}\right)$ is odd.

Therefore the integral is <u>zero</u> because we integrate from $-\frac{L}{2}$ to $\frac{L}{2}$.

So we only need to evaluate the case for $m=\text{odd}$:

$$a_m = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \underbrace{y(x')\,(-1)^{(m-1)/2} \cos\left(\frac{m\pi x'}{L}\right)}\,dx' \quad , \quad m=\text{odd}.$$

This integrand is <u>even</u> because $y(x')$ and $\cos\left(\frac{m\pi x'}{L}\right)$ are both even functions of x'. Since we integrate from $-\frac{L}{2}$ to $\frac{L}{2}$, we can just integrate from zero to $\frac{L}{2}$ and multiply by 2:

$$a_m = (2)\frac{2}{L}\int_{0}^{\frac{L}{2}} y(x')\,(-1)^{(m-1)/2} \cos\left(\frac{m\pi x'}{L}\right)dx'$$

$$a_m = (2)\left(\frac{2}{L}\right)(-1)^{(m-1)/2}\left(\frac{2h}{L}\right)\int_0^{\frac{L}{2}}\left(-x' + \frac{L}{2}\right)\cos\left(\frac{m\pi x'}{L}\right)dx'$$

$$= \left(\frac{8h}{L^2}\right)(-1)^{(m-1)/2}\left[\left(-\left(\frac{L}{m\pi}\right)^2\cos\left(\frac{m\pi x'}{L}\right) - \frac{x'L}{m\pi}\sin\left(\frac{m\pi x'}{L}\right) + \left(\frac{L}{2}\right)\left(\frac{L}{m\pi}\right)\sin\frac{m\pi x'}{L}\right)\Bigg|_0^{\frac{L}{2}}\right]$$

zero for m=odd

cancel

$$= \left(\frac{8h}{L^2}\right)(-1)^{(m-1)/2}\left[-\left(\frac{L}{m\pi}\right)^2\cos\left(\frac{m\pi}{2}\right) - \frac{L^2}{2m\pi}\sin\left(\frac{m\pi}{2}\right) + \left(\frac{L^2}{2m\pi}\right)\sin\left(\frac{m\pi}{2}\right) - \left(-\left(\frac{L}{m\pi}\right)^2\right)\right]$$

$$\boxed{a_m = \frac{8h}{(m\pi)^2}(-1)^{(m-1)/2}, \quad m = \text{odd}} \quad \text{and} \quad \boxed{a_m = \emptyset \text{ for } m \text{ even}}$$

Therefore

$$a_1 = \frac{8h}{\pi^2}$$

$$a_2 = \emptyset$$

$$a_3 = \frac{-8h}{9\pi^2}$$

$$a_4 = \emptyset$$

$$a_5 = \frac{8h}{25\pi^2}$$

$$\vdots$$

Or we can write

$$y(x, t=\phi) = \frac{8h}{\pi^2}\sin\left(\frac{\pi x}{L}\right) - \frac{8h}{9\pi^2}\sin\left(\frac{3\pi x}{L}\right) + \frac{8h}{25\pi^2}\sin\left(\frac{5\pi x}{L}\right.$$

$$+ \ldots$$

Why did we do this?

Recall our motivation: The general solution to the wave equation is a sum over normal modes.

$$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2} \implies y(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n t}$$

where c_n = real and imaginary
$\equiv a_n + i b_n$

Given the initial condition:

$$y(x, t=0) = \text{a triangle} = \begin{cases} \left(\frac{2h}{L}\right) x & , 0 \le x \le \frac{L}{2} \\ \frac{2h}{L}(L-x) & , \frac{L}{2} \le x \le L \end{cases}$$

We found the a_n:

$$a_n = \begin{cases} \left(\frac{8h}{(n\pi)^2}\right)(-1)^{(n-1)/2} & \text{for } n = \text{odd} \\ \phi & \text{for } n = \text{even} \end{cases}$$

What about the imaginary part, $\{b_n\}$?
It is determined by the <u>initial</u> <u>velocity</u>:

$$\dot{y}(x, t=0) = \sum_{n=1}^{\infty} -\omega_n b_n \sin\left(\frac{n\pi x}{L}\right)$$

If we release the string from rest, then we must have $\dot{y}(x, t=0) = 0 \Rightarrow \boxed{b_n = 0 \text{ for all } n}$

So our final, time-dependent solution is

$$\boxed{y(x,t) = \sum_{\substack{n=1 \\ \text{(only odd(n))}}}^{\infty} \frac{8h}{(n\pi)^2} (-1)^{\left(\frac{n-1}{2}\right)} \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n t}}$$ (odd n only!)

where $\omega_n = \sqrt{\frac{T}{\rho}} \frac{n\pi}{L}$, $n = 1, 2, 3, \ldots$

We write the initial condition function $y(x, t=0)$ as a sum over normal modes because then the time development is extremely simple: each normal mode goes forward in time with its own harmonic factor $\left(e^{i\omega_n t}\right)$.