Fourier's Trick relies upon the following mathematical identity:

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \begin{cases} 0, & m \neq n \\ \frac{L}{2}, & m = n \end{cases}$$

We like to write this more compactly:

Define $\boxed{\delta_{mn} \equiv \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}}$ "Kronecker Delta"

Thus $\delta_{11} = 1$, $\delta_{12} = \emptyset$, $\delta_{13} = \emptyset$, $\delta_{22} = 1$, ect

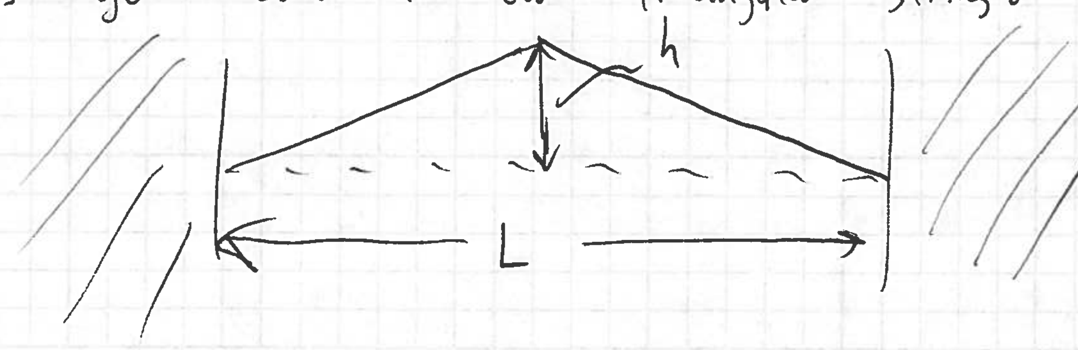
Using the Kronecker Delta we can say

$$\boxed{\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{L}{2}\delta_{mn}}$$

or

$$\boxed{\frac{2}{L}\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \delta_{mn}}$$

Let's go back to our triangular string:



This is the shape at $t = \emptyset$. The functional form is

$$y(x, t=\emptyset) = \begin{cases} \left(\frac{2h}{L}\right) x & , \quad 0 \le x \le \frac{L}{2} \\ \left(\frac{2h}{L}\right)(L-x) & , \quad \frac{L}{2} \le x \le L \end{cases}$$

We want to describe this simple function in a much more complicated way: as an infinite sum of normal modes:

$$y(x, t=\emptyset) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

The question is: what are the $\{a_n\}$?

Fourier's Trick tells us that any particular coefficient, ~~can be calculated~~ for example, the m^{th} coefficient (a_m), can be calculated by evaluating this integral:

$$\boxed{a_m = \frac{2}{L}\int_0^L \sin\left(\frac{m\pi x}{L}\right) y(x)\, dx}$$

For our function $y(x)$, this integral is

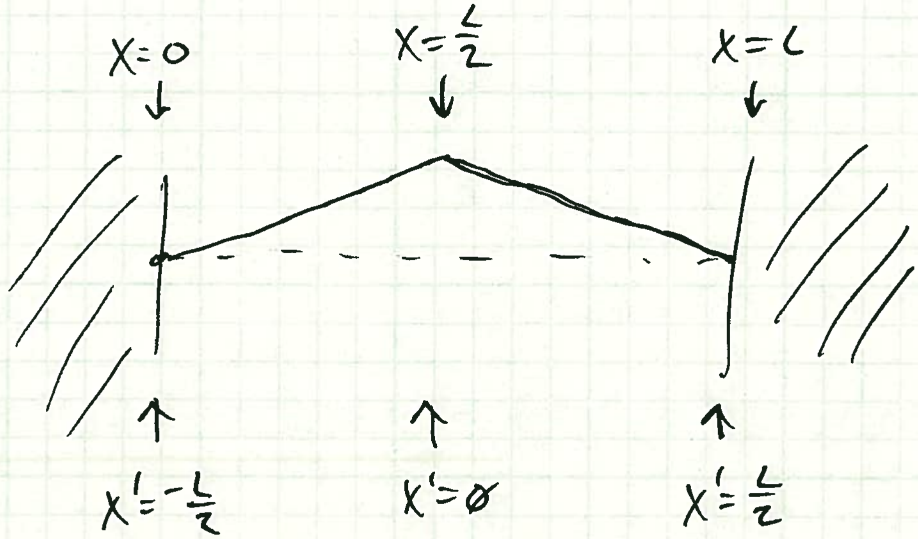
$$a_m = \frac{2}{L}\int_0^{L/2} \sin\left(\frac{m\pi x}{L}\right)\left(\frac{2hx}{L}\right)dx + \frac{2}{L}\int_{\frac{L}{2}}^{L} \sin\left(\frac{m\pi x}{L}\right)\left(\frac{2h(L-x)}{L}\right)dx$$

It turns out that the easiest way to evaluate this integral is to move our coordinate system -

$$\text{Let} \quad x' \equiv x - \frac{L}{2}$$

$$\text{so that} \quad x = x' + \frac{L}{2}$$

This means that $x' = \emptyset$ is the center of the string



In terms of x', our string position at $t = \emptyset$ is

$$\boxed{y(x', t=\emptyset) = \begin{cases} \left(\dfrac{2h}{L}\right)\left(x' + \dfrac{L}{2}\right), & -\dfrac{L}{2} \le x' \le \emptyset \\[2ex] \left(\dfrac{2h}{L}\right)\left(-x' + \dfrac{L}{2}\right), & 0 \le x' \le \dfrac{L}{2} \end{cases}}$$

Note that y is an even function of x'.

Also, we have the following math theorem:

If $\quad x = x' + \dfrac{L}{2}$,

Then $\quad \sin\left(\dfrac{m\pi x}{L}\right) = \begin{cases} (-1)^{(m-1)/2} \cos\left(\dfrac{m\pi x'}{L}\right), & \text{for } m = \text{odd} \\[2ex] (-1)^{m/2} \sin\left(\dfrac{m\pi x'}{L}\right), & \text{for } m = \text{even} \end{cases}$

Now our integral has 2 cases:

$$a_m = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} y(x')\,(-1)^{(m-1)/2} \cos\left(\frac{m\pi x'}{L}\right)dx' \quad \text{for } m = \text{odd}$$

AND

$$a_m = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \underbrace{y(x')\,(-1)^{(m-2)/2} \sin\left(\frac{m\pi x'}{L}\right)}\,dx' \quad \text{for } m = \text{even}$$

This integrand is an ~~even~~ odd function of x', because $y(x')$ is even, and $\sin\left(\frac{m\pi x'}{L}\right)$ is odd.

Therefore the integral is zero because we integrate from $-\frac{L}{2}$ to $\frac{L}{2}$.

So we only need to evaluate the case for $m = \text{odd}$:

$$a_m = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \underbrace{y(x')\,(-1)^{(m-1)/2} \cos\left(\frac{m\pi x'}{L}\right)}\,dx' \quad , \ m = \text{odd}.$$

This integrand is even because $y(x')$ and $\cos\left(\frac{m\pi x'}{L}\right)$ are both even functions of x'. Since we integrate from $-\frac{L}{2}$ to $\frac{L}{2}$, we can just integrate from zero to $\frac{L}{2}$ and multiply by 2:

$$a_m = (2)\frac{2}{L}\int_{0}^{\frac{L}{2}} y(x')\,(-1)^{(m-1)/2} \cos\left(\frac{m\pi x'}{L}\right)dx'$$

$$a_m = (2)\left(\frac{2}{L}\right)(-1)^{(m-1)/2}\left(\frac{2h}{L}\right)\int_0^{\frac{L}{2}}\left(-x' + \frac{L}{2}\right)\cos\left(\frac{m\pi x'}{L}\right)dx'$$

$$= \left(\frac{8h}{L^2}\right)(-1)^{(m-1)/2}\left[\left(-\left(\frac{L}{m\pi}\right)^2\cos\left(\frac{m\pi x'}{L}\right) - \frac{x'L}{m\pi}\sin\left(\frac{m\pi x'}{L}\right) + \left(\frac{L}{2}\right)\left(\frac{L}{m\pi}\right)\sin\frac{m\pi x'}{L}\right)\Bigg|_0^{\frac{L}{2}}\right]$$

zero for m=odd

cancel

$$= \left(\frac{8h}{L^2}\right)(-1)^{(m-1)/2}\left[-\left(\frac{L}{m\pi}\right)^2\cos\left(\frac{m\pi}{2}\right) - \frac{L^2}{2m\pi}\sin\left(\frac{m\pi}{2}\right) + \left(\frac{L^2}{2m\pi}\right)\sin\left(\frac{m\pi}{2}\right) - \left(-\left(\frac{L}{m\pi}\right)^2\right)\right]$$

$$\boxed{a_m = \frac{8h}{(m\pi)^2}(-1)^{(m-1)/2}, \quad m = \text{odd}} \quad \text{and} \quad \boxed{a_m = \phi \text{ for } m \text{ even}}$$

Therefore

$$a_1 = \frac{8h}{\pi^2}$$

$$a_2 = \phi$$

$$a_3 = \frac{-8h}{9\pi^2}$$

$$a_4 = \phi$$

$$a_5 = \frac{8h}{25\pi^2}$$

$$\vdots$$

Or we can write

$$\cancel{}\ y(x, t=\emptyset) = \frac{8h}{\pi^2}\sin\left(\frac{\pi x}{L}\right) - \frac{8h}{9\pi^2}\sin\left(\frac{3\pi x}{L}\right) + \frac{8h}{25\pi^2}\sin\left(\frac{5\pi x}{L}\right.$$

$$+ \ldots$$

Why did we do this?

Recall our motivation: The general solution to the wave equation is a sum over normal modes.

$$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2} \implies y(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n t}$$

where c_n = real and imaginary
$\equiv a_n + i b_n$

Given the initial condition:

$$y(x, \cancel{t=0}) = \text{a triangle} = \begin{cases} \left(\frac{2h}{L}\right) x & , 0 \le x \le \frac{L}{2} \\ \frac{2h}{L}(L-x) & , \frac{L}{2} \le x \le L \end{cases}$$

We found the a_n:

$$a_n = \begin{cases} \left(\frac{8h}{(n\pi)^2}\right)(-1)^{(n-1)/2} & \text{for } n = \text{odd} \\ \emptyset & \text{for } n = \text{even} \end{cases}$$

What about the imaginary part, $\{b_n\}$?
It is determined by the <u>initial</u> <u>velocity</u>:

$$\dot{y}(x, t=0) = \sum_{n=1}^{\infty} -\omega_n b_n \sin\left(\frac{n\pi x}{L}\right)$$

If we release the string from rest, then we must have $\dot{y}(x, t=0) = 0 \Rightarrow \boxed{b_n = 0 \text{ for all } n}$

So our final, time-dependent solution is

$$\boxed{y(x,t) = \sum_{\substack{n=1 \\ (\text{only odd}(n))}}^{\infty} \frac{8h}{(n\pi)^2} (-1)^{\left(\frac{n-1}{2}\right)} \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n t}}$$ (odd n only!)

where $\omega_n = \sqrt{\frac{T}{\rho}} \frac{n\pi}{L}$, $n = 1, 2, 3, \ldots$

We write the initial condition function $y(x, t=0)$ as a sum over normal modes because then the time development is extremely simple: each normal mode goes forward in time with its own harmonic factor $\left(e^{i\omega_n t}\right)$.

Orthogonal Functions

We say that the set of functions $\left\{\sin \frac{n\pi x}{L}\right\}$, $n=1,2,3,4,\ldots$ are **orthogonal** because they have the following property:

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0, & m \neq n \\ \frac{L}{2}, & m = n \end{cases}$$

Why does this mean they are "orthogonal"? Why do we use that word?

Answer: These functions are actually vectors, and they are perpindicular to each other just like vectors can be perpindicular to each other.

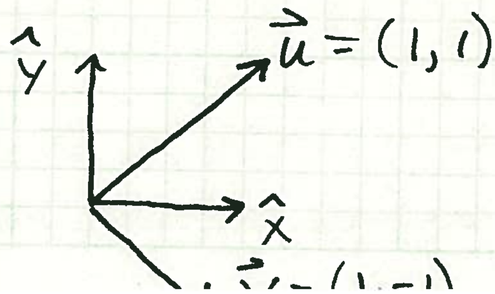
Consider 2-dimensional vectors:

$\hat{x} = (1, \emptyset)$ $\quad \hat{y} = (\emptyset, 1)$

These are orthogonal because their dot product is zero:

$$\hat{x}\cdot\hat{y} = (1,\emptyset)\cdot(\emptyset,1) = (1)(\emptyset) + (\emptyset)(1) = \emptyset$$

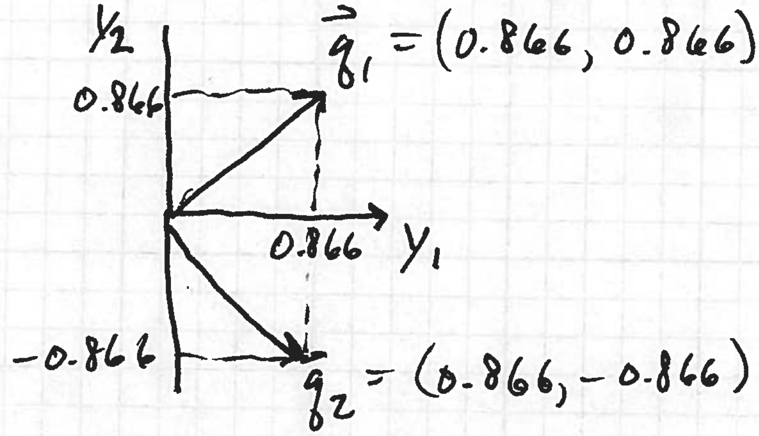
How about these vectors?



Are $\vec{u}$ and $\vec{v}$ orthogonal?

Yes: $\vec{u}\cdot\vec{v} = (1,1)\cdot(1,-1)$
$= 1 - 1 = \emptyset$ ✓

How about these vectors:



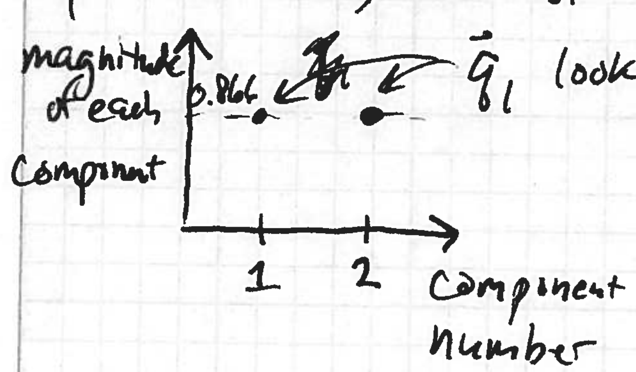
Are $\vec{q}_1$ and $\vec{q}_2$ orthogonal?

Answer: Yes, as we know from Homework #7:

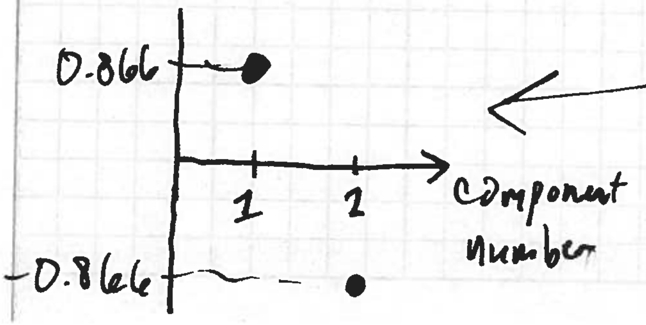
$$\vec{q}_1 \cdot \vec{q}_2 = (0.866, 0.866) \cdot (0.866, -0.866)$$

$$= \emptyset \quad \checkmark \text{ othogonal}$$

Now let's draw these vectors differently, in a very funny way: On the x-axis, let's put the component #, and on the y-axis let's put the magnitude of that component:



$\vec{q}_1$ looks like this



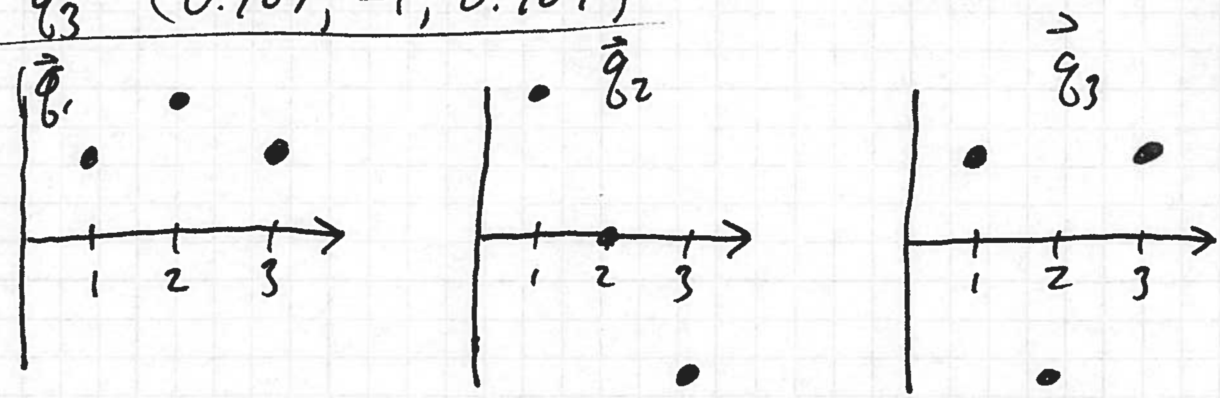
$\vec{q}_2$ looks like this

We can see that $\vec{q}_1$ & $\vec{q}_2$ are orthogonal because if we multiply and ~~add~~ sum the two components, the result is zero, because the second term cancels the first.

Let's Draw the 3 normal mode vectors for the 3-mass loaded string this way:

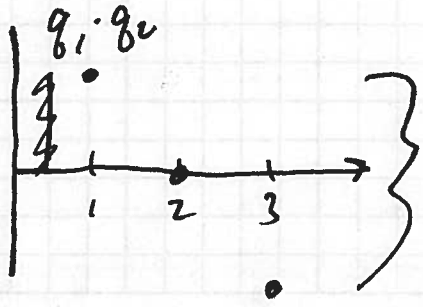
Recall from Homework #7:

$$\vec{q}_1 = (0.707, 1, 0.707)$$
$$\vec{q}_2 = (1, 0, -1)$$
$$\vec{q}_3 = (0.707, -1, 0.707)$$



To take a dot product, we would multiply each component and sum over all components.

Dot product of $\vec{q}_1$ and $\vec{q}_2$:



sum all 3 and get zero.

How about the 4-mass loaded string?

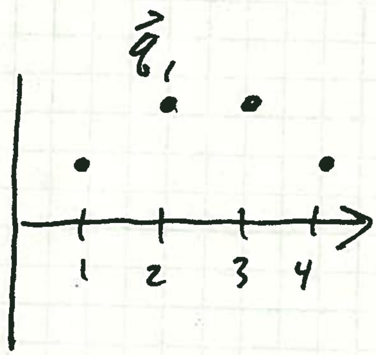
Again, from Homework #7,

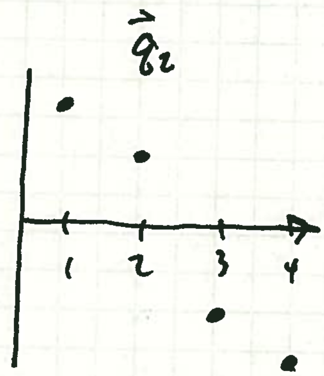
$$\vec{q}_1 = (0.588, 0.951, 0.951, 0.588)$$

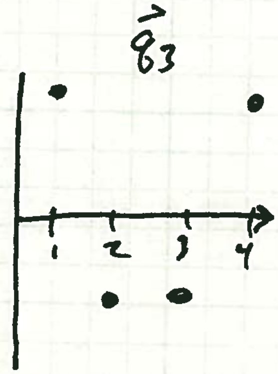
$$\vec{q}_2 = (0.951, 0.588, -0.588, -0.951)$$

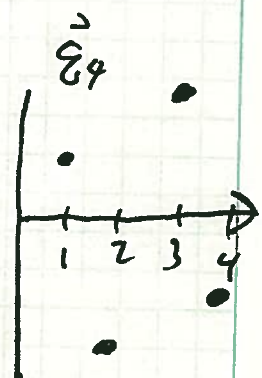
$$\vec{q}_3 = (0.951, -0.588, -0.588, 0.951)$$

$$\vec{q}_4 = (0.588, -0.951, 0.951, -0.588)$$



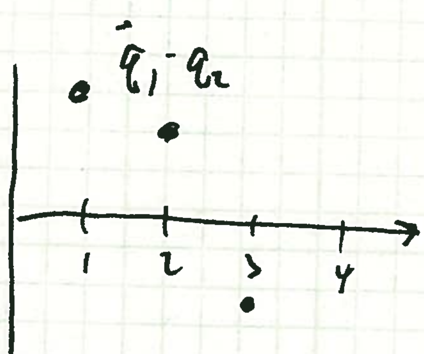






Can we see that $\vec{q}_1$ is orthogonal to $\vec{q}_2$?

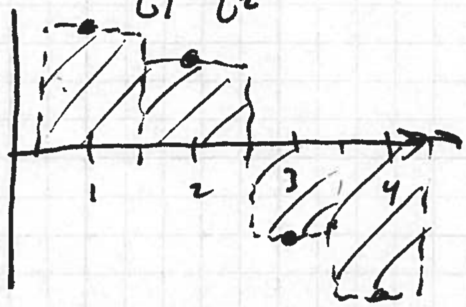
Yes! If we multiply components and sum over all components, we can see that the result will be zero, because the first two components get cancelled by the second two.



sum over all four

How can we visualize the sum? As an integral, in a discrete form

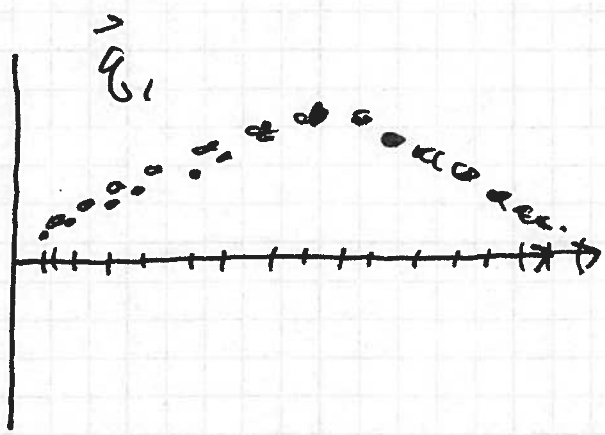
$\vec{q}_1 \cdot \vec{q}_2$

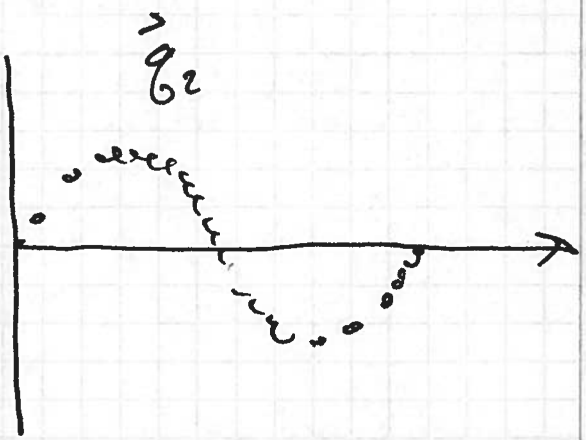


The area is zero, which means the dot product $\vec{q}_1 \cdot \vec{q}_2 = \emptyset$.

Now consider a loaded string with N = some large number of masses.

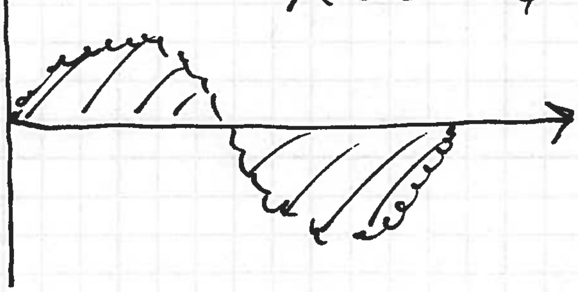
Let's draw the first and second normal modes





Multiply and sum to get the dot product

Area $= \emptyset$, so $\vec{q}_1 \cdot \vec{q}_2 = \emptyset$.



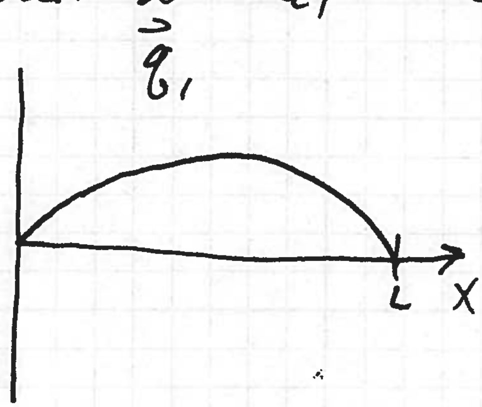
Now let the number of masses go to <u>infinity</u>:

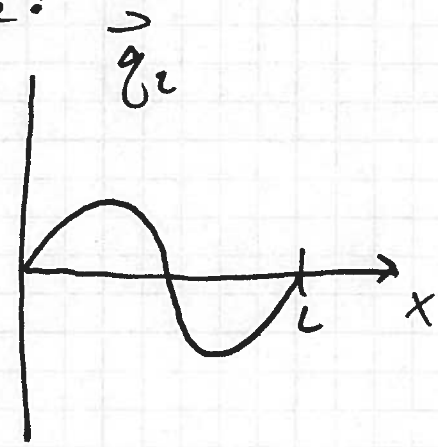
This is the case of the continuous string.
Now the normal modes are smooth functions of x:

normal mode 1 $= \sin\left(\frac{\pi x}{L}\right)$ ← A vector with an infinite # of components

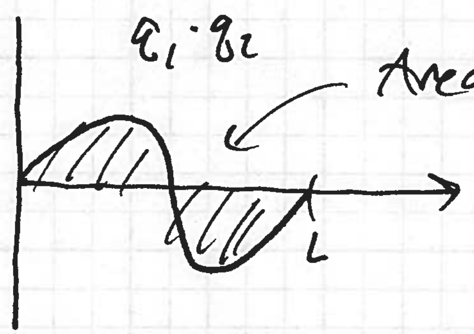
normal mode 2 $= \sin\left(\frac{2\pi x}{L}\right)$

What do they look like?





Are they orthogonal? To find out, just multiply and integrate:



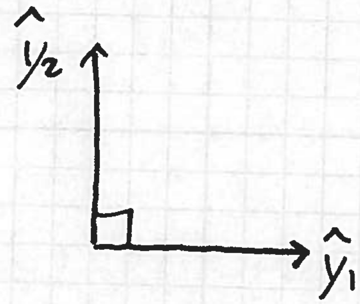
Area $= \emptyset$, so they are <u>orthogonal</u>.

This is what we mean when we say

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \emptyset$$

More on Orthogonal Functions & Vectors

Consider 2-dimensional vectors: $\vec{y} = (y_1, y_2)$



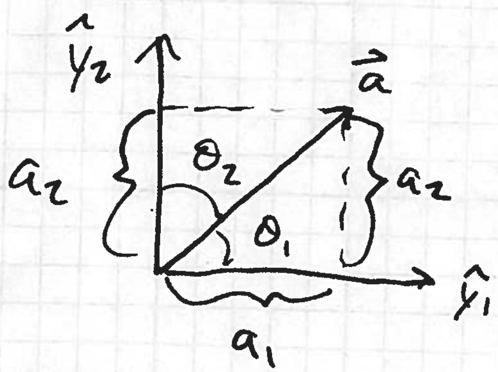
$\hat{y}_1$ & $\hat{y}_2$ are orthogonal unit vectors.

Therefore
$$\hat{y}_1 \cdot \hat{y}_2 = \emptyset$$
$$\hat{x}_1 \cdot \hat{y}_1 = 1$$
$$\hat{y}_2 \cdot \hat{y}_2 = 1$$

Summarizing:

orthogonality & normal length condition. $\longrightarrow$ $\boxed{\hat{y}_i \cdot \hat{y}_j = \delta_{ij}}$

We can write an arbitrary vector $\vec{a}$ as a sum of $\hat{y}_1$ & $\hat{y}_2$:



$$\vec{a} = a_1 \hat{y}_1 + a_2 \hat{y}_2 = \sum_{j=1}^{2} a_j \hat{y}_j$$

From Geometry, we know that

$$a_1 = |\vec{a}| \cos\theta_1$$
$$a_2 = |\vec{a}| \cos\theta_2$$

But we also know that

$$\vec{a} \cdot \hat{y}_1 = |\vec{a}| \underbrace{|\hat{x}_1|}_{1} \cos\theta_1 = |\vec{a}| \cos\theta_1 = a_1$$

and $\vec{a}\cdot\hat{y}_2 = |\vec{a}|\underbrace{|\hat{y}_2|}_{1}\cos\theta_2 = |\vec{a}|\cos\theta_2 = a_2$

$$\therefore \quad a_1 = \vec{a}\cdot\hat{y}_1, \quad a_2 = \vec{a}\cdot\hat{y}_2 \qquad \text{or} \qquad \boxed{a_i = \vec{a}\cdot\hat{y}_i}$$

This says: if you want to know the 1st component of $\vec{a}$, you should take the dot product with $\hat{y}_1$. If you want to know the second component of $\vec{a}$, you should take the dot product with $\hat{y}_2$.

We can get this result with algebra alone:

$$\vec{a}\cdot\hat{y}_i = \left(\sum_{j=1}^{2} a_j\hat{y}_j\right)\cdot\hat{y}_i = \sum_{j=1}^{2} a_j \underbrace{(\hat{y}_j\cdot\hat{y}_i)}_{\delta_{ij}}$$

$$= \sum_{j=1}^{2} a_j\,\delta_{ij}$$

Kronecker Delta kills all terms in the sum except $i=j$

$$= \cancel{a_1\delta_{i1} + a_2\delta_{i2}}$$

$$= a_i$$

$$\therefore \boxed{a_i = \vec{a}\cdot\hat{y}_i}$$

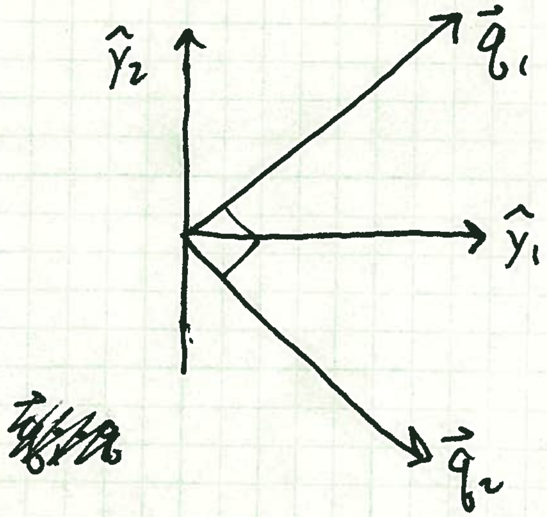
← same result, but no geometrical argument. (Purely algebra)

Now consider the 2 normal modes of the 2-mass loaded string:

$$\vec{q}_1 = (1,1) \quad \text{and} \quad \vec{q}_2 = (1,-1)$$

We can draw these as

$$\vec{q}_1 \cdot \vec{q}_2 = \emptyset$$



We know the general solution for this system is

$$\vec{y}(t) = (y_1(t), y_2(t))$$
$$= \sum_{n=1}^{2} c_n \vec{q}_n e^{i\omega_n t}$$

Our job is to figure out what c_1 & c_2 are, given a particular initial condition.

c_1 & c_2 are complex: $c_n = a_n + i b_n$, but if the initial ~~xxxx~~ velocity is zero, then b_1 & b_2 will be zero:

$$\vec{y}(t) = \sum_{n=1}^{2} a_n \vec{q}_n e^{i\omega_n t}$$

(for initial velocity $= \emptyset$ ($\dot{y}_1(\emptyset) = \emptyset$, $\dot{y}_2(\emptyset) = \emptyset$).)

Now our job is to figure out what a_1 & a_2 are, given an initial position

$$\vec{y}_0 = \vec{y}(t=\emptyset) = (y_{10}, y_{20}) = \sum_{n=1}^{2} a_n \vec{q}_n$$

his is how we do it: take the dot product of our initial position vector with each normal mode vector:

$$\vec{y}_0 \cdot \vec{q}_1 = \left(\sum_{n=1}^{2} a_n \vec{q}_n\right) \cdot \vec{q}_1 = \sum_{n=1}^{2} a_n (\vec{q}_n \cdot \vec{q}_1)$$

$$= a_1\, \vec{q}_1 \cdot \vec{q}_1 + a_2 \cancelto{0}{\vec{q}_2 \cdot \vec{q}_1}$$

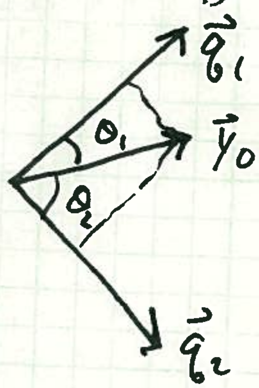
$$= a_1 |\vec{q}_1|^2$$

$$\therefore \quad \boxed{a_1 = \frac{\vec{y}_0 \cdot \vec{q}_1}{|\vec{q}_1|^2}}$$

Similarly,

$$\boxed{a_2 = \frac{\vec{y}_0 \cdot \vec{q}_2}{|\vec{q}_2|^2}}$$

Geometrically,



$\vec{y}_0 \leftarrow$ some arbitrary initial position

$$\vec{y}_0 = \left(\frac{\vec{y}_0 \cdot \vec{q}_1}{|\vec{q}_1|^2}\right) \vec{q}_1 + \left(\frac{\vec{y}_0 \cdot \vec{q}_2}{|\vec{q}_2|^2}\right) \vec{q}_2$$

$$\vec{y}_0 \cdot \vec{q}_1 = |\vec{y}_0||\vec{q}_1| \cos\theta_1$$

$$\vec{y}_0 \cdot \vec{q}_2 = |\vec{y}_0||\vec{q}_2| \cos\theta_2$$

$$\vec{y}_0 = |\vec{y}_0| \cos\theta_1 \frac{\vec{q}_1}{|\vec{q}_1|} + |\vec{y}_0| \cos\theta_2 \frac{\vec{q}_2}{|\vec{q}_2|}$$

initial condition

Summarizing,

$$\boxed{a_i = \frac{\vec{y}_0 \cdot \vec{q}_i}{|\vec{q}_i|^2}}$$

To calculate the i^{th} expansion coefficient, take the dot product of the initial condition with each normal mode vector

a "normalization factor"

We have to divide by $|\vec{q}_i|^2$ because it does not have length = 1:

$$\vec{q}_1 = (1, 1) \Rightarrow |\vec{q}_1|^2 = 2$$

$$\vec{q}_2 = (1, -1) \Rightarrow |\vec{q}_2|^2 = 2$$

We should call this rule "Fourier's Trick for N masses", because this is exactly what Fourier's Trick does:

$$a_n = \left(\frac{2}{L}\right) \int_0^L \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{\substack{\text{a normal} \\ \text{mode vector}}} \underbrace{y(x, t=0)}_{\substack{\text{the initial} \\ \text{condition}}} \, dx$$

the integral is a dot product between $\sin\left(\frac{n\pi x}{L}\right)$ and $y(x, t=0)$

normalization factor.

Just like $a_i = \dfrac{\vec{q}_i \cdot \vec{y}_0}{|\vec{q}_i|^2}$

We cannot show Fourier's Trick geometrically, because ~~I really draw~~ these "continuous vectors" have an infinite number of components, and I can't draw any more than 2 orthogonal directions.

But we can do the algebraic proof:

$$y(x,t=0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$ ← correct as long as initial velocity is zero

What is a_n?

Answer: Take the dot product with one particular function (vector): $\sin\left(\frac{m\pi x}{L}\right)$:

$$\underbrace{\int_0^L y(x,t=0)\sin\left(\frac{m\pi x}{L}\right)dx}_{\text{dot product}} =$$

$$= \int_0^L\left(\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

$$= \sum_{n=1}^{\infty} a_n \underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx}_{\frac{L}{2}\delta_{nm}}$$

because these vectors are orthogonal

$$= \sum_{n=1}^{\infty} a_n \delta_{nm}\left(\frac{L}{2}\right) = a_m\left(\frac{L}{2}\right)$$

Kills all terms except $n=m$

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$$\therefore \boxed{a_m = \frac{2}{L}\underbrace{\int_0^L y(x,t=\emptyset)\sin\left(\frac{m\pi x}{L}\right)dx}_{\text{a normalized dot product.}}}$$

It's exactly the same proof as

$$a_i = \frac{\vec{Y}_0 \cdot \vec{q}_i}{|\vec{q}_i|^2}$$