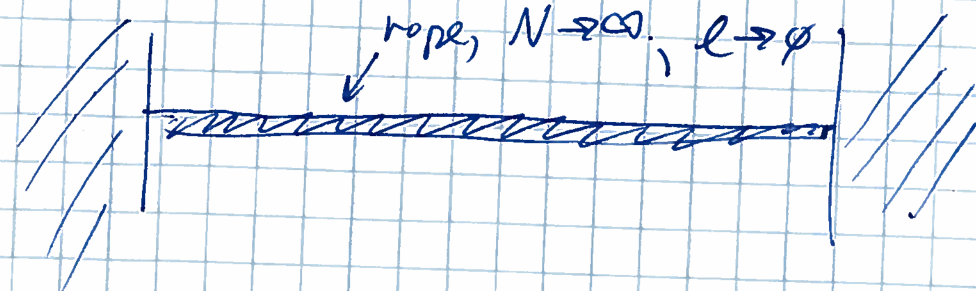
Continuous Systems — Wave Equation

We can model a continuous system, like a rope, as being a limit where the number of particles goes to infinity and ℓ goes to zero.



For N masses, our equation of motion was

$$\ddot{y}_p + 2\omega_0^2 y_p - \omega_0^2(y_{p+1} + y_{p-1}) = \emptyset$$

or

$$\ddot{y}_p = \omega_0^2(y_{p+1} - y_p) - \omega_0^2(y_p - y_{p-1})$$

Recall $\omega_0^2 = \frac{T}{m\ell}$

$$\therefore \quad m\ddot{y}_p = \frac{T}{\ell}(y_{p+1} - y_p) - \frac{T}{\ell}(y_p - y_{p-1})$$

Divide by ℓ:

$$\frac{m}{\ell}\ddot{y}_p = \frac{T}{\ell}\left(\frac{y_{p+1} - y_p}{\ell}\right) - \frac{T}{\ell}\left(\frac{y_p - y_{p-1}}{\ell}\right)$$

As ℓ goes to zero:

$$\lim_{\ell \to \emptyset}\left(\frac{y_{p+1} - y_p}{\ell}\right) \Rightarrow \lim_{\ell \to \emptyset}\left(\frac{y(x+\ell) - y(x)}{\ell}\right) = \left.\frac{dy}{dx}\right|_{x+\frac{\ell}{2}}$$

$$\lim_{\ell \to \emptyset}\left(\frac{y_p - y_{p-1}}{\ell}\right) \Rightarrow \lim_{\ell \to \emptyset}\left(\frac{y(x) - y(x-\ell)}{\ell}\right) = \left.\frac{dy}{dx}\right|_{x-\frac{\ell}{2}}$$

Also, let $\frac{m}{\ell} = \rho$ = mass density per unit length

Then

$$\cancel{\rho}\ \rho \ddot{y}(x) = \frac{T}{\ell}\left[\frac{dy}{dx}\Big|_{x+\frac{\ell}{2}} - \frac{dy}{dx}\Big|_{x-\frac{\ell}{2}}\right]$$

$$\cancel{\rho}\ \rho \frac{d^2y}{dt^2} = T \lim_{\ell\to 0} \underbrace{\left[\frac{\frac{dy}{dx}\Big|_{x+\frac{\ell}{2}} - \frac{dy}{dx}\Big|_{x-\frac{\ell}{2}}}{\ell}\right]}_{\frac{d^2y}{dx^2}}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{\rho}{T}\frac{d^2y}{dt^2}}$$ "Wave Equation"

This is the Eq. of Motion for a continuous system of masses. It is Newton's 2nd Law.

Solution: The normal modes we can get by allowing $N \to \infty$ in the N-mass system, while $\ell \to 0$ such that $(N+1)\ell = L$ = total length

For N particles,

$$y_{pn} = C_n \sin\left(\frac{pn\pi}{N+1}\right)$$ Now $p\ell = x$ = distance along rope

20

i.e. $y_n(x) = C_n \sin\left(\frac{\overbrace{(p\ell)}^{x}\pi n}{\underbrace{\ell(N+1)}_{L = \text{total length}}}\right) = \boxed{C_n \sin\left(\frac{n\pi x}{L}\right)}$ $n = 1, 2, \ldots \infty$

Amplitude relationship for normal modes of of continuous system

The frequencies are

$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right)$$

$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi\ell}{2\underbrace{(N+1)\ell}_{L}}\right) = 2\omega_0 \sin\left(\frac{n\pi\ell}{2L}\right)$$

In the limit when $\ell \to 0$, $\sin\left(\frac{n\pi\ell}{2L}\right) \to \frac{n\pi\ell}{2L}$

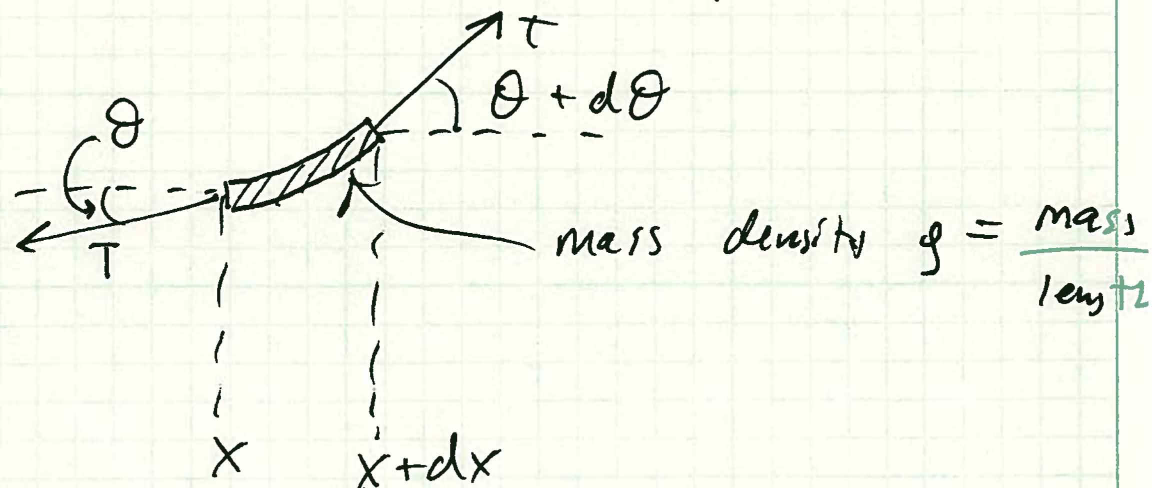
$$\omega_n = 2\omega_0\left(\frac{n\pi\ell}{2L}\right)$$

$$\omega_0 = \sqrt{\frac{T}{m\ell}} = \sqrt{\frac{T/\ell^2}{m\ell}} = \frac{1}{\ell}\sqrt{\frac{T}{\rho}}, \quad \rho = \frac{m}{\ell}$$

$$\boxed{\omega_n = \sqrt{\frac{T}{\rho}}\ \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots \infty}$$

Another Derivation of the Wave Equation.

Consider a short segment of rope:



The Force on the segment of rope has two components:

$$F_y = T\sin(\theta + d\theta) - T\sin(\theta)$$
$$F_x = T\cos(\theta + d\theta) - T\cos(\theta)$$

For small $d\theta$, $\sin(\theta + d\theta) \approx \sin\theta + d\theta$
$\cos(\theta + d\theta) \approx \cos\theta$

So $F_y \approx T\,d\theta$
$F_x \approx \emptyset$

So the Eq. of Motion in the y direction is

$$T\,d\theta = (m)\ddot{y} = (\rho\,dx)\ddot{y}$$

Also $\tan\theta = \frac{\partial y}{\partial x}$ ← take derivative w/respect to θ

$$\frac{d}{d\theta}(\tan\theta) = \frac{d}{d\theta}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial^2 y}{\partial x^2}\frac{\partial x}{\partial \theta}$$

$$\sec^2\theta = \frac{\partial^2 y}{\partial x^2}\frac{\partial x}{\partial \theta}$$

$\sec^2\theta \approx 1$ because θ is small

$$\therefore \quad 1 = \frac{\partial^2 y}{\partial x^2}\frac{\partial x}{\partial \theta}$$

$$d\theta = \frac{\partial^2 y}{\partial x^2}\,dx$$

So we have

$$T\,d\theta = (\rho\,dx)\,\ddot{y}$$

$$T\left(\frac{\partial^2 y}{\partial x^2}\,dx\right) = (\rho\,dx)\,\ddot{y}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}\ddot{y} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2}$$

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2}}$$

$\Leftarrow$ Equation of Motion for a continuous string

"Wave Equation.

Summarizing

Equation of Motion: $\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T} \frac{\partial^2 y}{\partial t^2}$

Normal Mode Amplitude Relationship: $y_n(x) = C_n \sin\left(\frac{n\pi x}{L}\right)$

Normal Mode Frequencies: $\omega_n = \sqrt{\frac{T}{\rho}} \frac{n\pi}{L}$, $n = 1, 2, 3, \ldots \infty$

General Solution:

$$y(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n t}$$

C_n = complex (real & imaginary parts)
(or amplitude & phase)

Required Initial conditions: $\begin{cases} y(x, t=0) = ? \\ \dot{y}(x, t=0) = ? \end{cases}$

These determine the real & imaginary parts of C_n