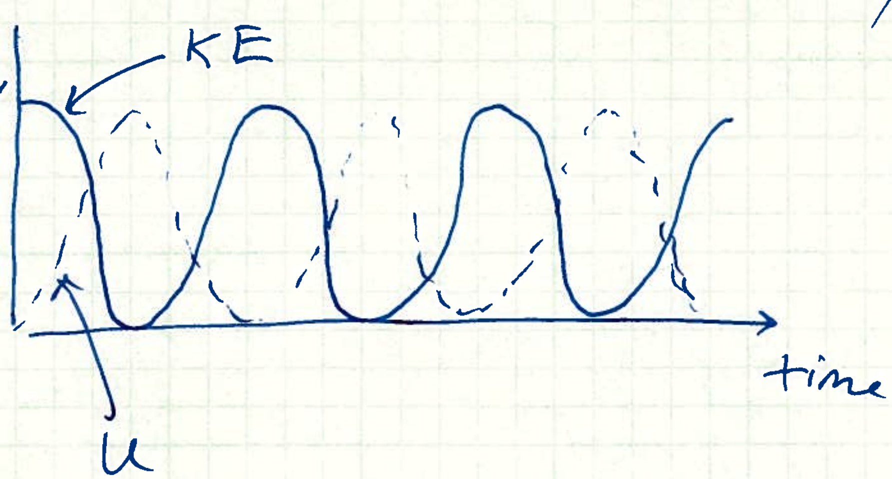
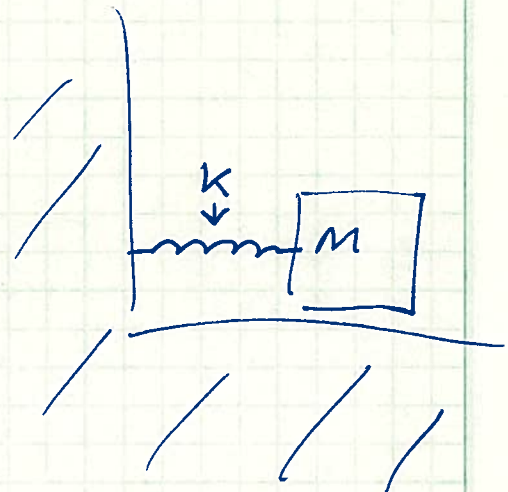
Electrical Oscillators

Mechanical Oscillator: energy is converted between kinetic and elastic potential

$KE = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$

$U = \frac{1}{2}kx^2$

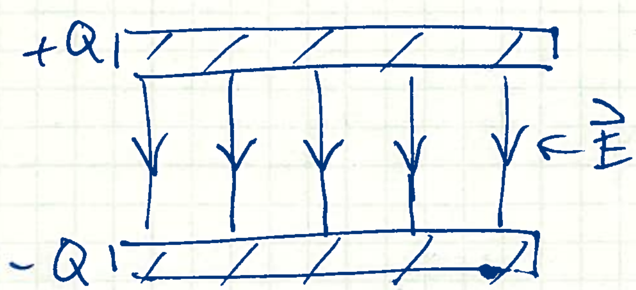




Electrical Oscillator: Energy is converted between electric ($\vec{E}$ field) and magnetic ($\vec{B}$ field).

(1) Capacitor: Device for storing energy in an electric field

Ex: Parallel Plate Capacitor



Each small volume of space (dV) with an electric field $\vec{E}$ stores a small amount of electric energy (dU_E):

$$dU_E = \underbrace{\frac{1}{2}\varepsilon_0 |\vec{E}|^2}_{u_E} dV \quad , \quad u_E \equiv \frac{1}{2}\varepsilon_0 |\vec{E}|^2$$

Energy, big U

u_E ↑ energy density, little u

= "electric energy density of free space"

$= \frac{\text{Joules}}{\text{meter}^3}$

The total energy stored is

"big U" $\rightarrow U_E = \int_{\text{all space}} \frac{1}{2}\varepsilon_0 |\vec{E}|^2 dV$

If the electric field is created by a capacitor with charges $+Q$ and $-Q$ and voltage difference V, then the total energy can be written

$$U_E = \frac{1}{2}QV = \frac{1}{2}CV^2 = \frac{Q^2}{2C}$$

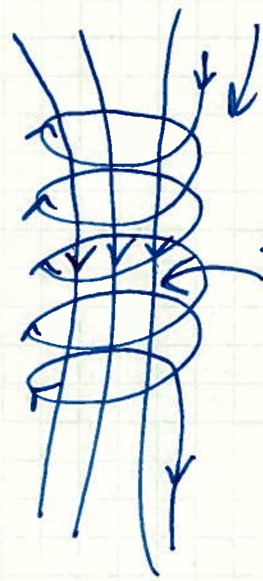
where C = capacitance $= \frac{Q}{V}$

Capacitance is a constant that only depends on the shape and material of the capacitor.

"$Q = CV$" says "charge on the capacitor is proportional to the voltage across it. C is the proportionality constant."

(2) Inductor: Device for storing energy in a magnetic field.

A simple solenoid:



The energy density of a magnetic field is

$$dU_B = \frac{1}{2\mu_0}|\vec{B}|^2 dV \quad , \quad u_B = \frac{1}{2\mu_0}|\vec{B}|^2$$

"little u" = "magnetic energy density of free space"

= $\frac{\text{Joules}}{\text{meter}^3}$

The total magnetic energy is

"big U" $\rightarrow$ $$U_B = \int_{\text{all space}} \frac{1}{2\mu_0}|\vec{B}|^2 dV$$

For an inductor, the total energy can be written as

$$U_B = \frac{1}{2}LI^2 \quad , \text{ where } L = \text{"self-inductance"}$$

L is determined by the shape and material of the inductor. It is the proportionality constant between magnetic flux and current:

Φ_B = magnetic flux through the inductor $= L\,I$

L = proportionality constant. I = current

Similarities between C & L

Device	Circuit Symbol	MKS unit	Stores energy in:	Proportionality Constant:	Determined by
C	capacitor symbol	Farad	$\vec{E}$	$Q = CV$	Shape and material
L	inductor coil symbol	Henry	$\vec{B}$	$\Phi_B = LI$	Shape and material

If you know the shape & material of your {capacitor / inductor}, then you can calculate {C / L}.

Neither C nor L depends on: {Q, V, I} These things depend on time, but C & L are constant.

Voltage Rules:

Capacitor: $|V_c| = \left|\frac{1}{C}Q\right|$ or $\boxed{V_C = \frac{1}{C}Q}$ (ignoring any sign)

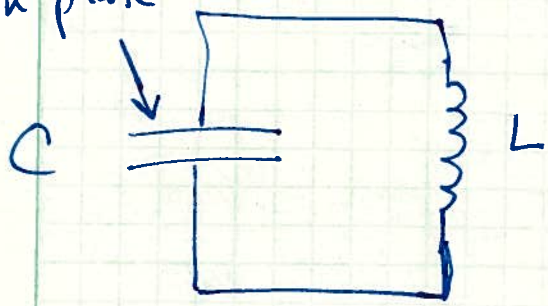
Inductor: $|V_L| = \left|-\frac{d\Phi_B}{dt}\right| = \left|-\frac{d}{dt}(LI)\right| = \left|L\frac{dI}{dt}\right|$

$\therefore \boxed{V_L = L\frac{dI}{dt}}$ (ignoring any sign)

LC oscillator

Simplest electrical oscillator.
Energy exchanges between electric & magnetic.

q = charge on plate



Voltage Loop rule:

$$V_c + V_L = \emptyset$$

$$\left(\frac{1}{C} q\right) \left(L \frac{dI}{dt}\right) = L \frac{d^2 q}{dt^2}$$

since $I = \frac{dq}{dt}$

$$\therefore \quad \frac{1}{C} q + L \frac{d^2 q}{dt^2} = \emptyset$$

$$\boxed{\ddot{q} + \frac{1}{LC} q = \emptyset}$$ Simple harmonic oscillator equation

Solution: $q(t) = q_0 e^{i(\omega_0 t + \delta)}$, where $\omega_0 = \frac{1}{\sqrt{LC}}$ = "natural freq."

q_0 & δ are determined by the initial conditions

$$\dot{q}(t) = I(t) = (i\omega_0)\left(q_0 e^{i(\omega_0 t + \delta)}\right)$$
$$= i\omega_0 q(t)$$

current has a phase shift of 90° compared to charge c

Energy: U_E = electric energy

$$= \frac{1}{2C} q^2 = \frac{1}{2C}\left[q_0 \cos(\omega_0 t + \delta)\right]^2$$

$$= \frac{q_0^2}{2C} \cos^2(\omega_0 t + \delta)$$

U_B = magnetic energy

$$= \frac{1}{2} L I^2$$

$$= \frac{1}{2} L \left[\underbrace{\mathrm{Re}\left(i\omega_0 q_0 e^{i(\omega_0 t + \delta)}\right)}_{-\omega_0 q_0 \sin(\omega_0 t + \delta)} \right]^2$$

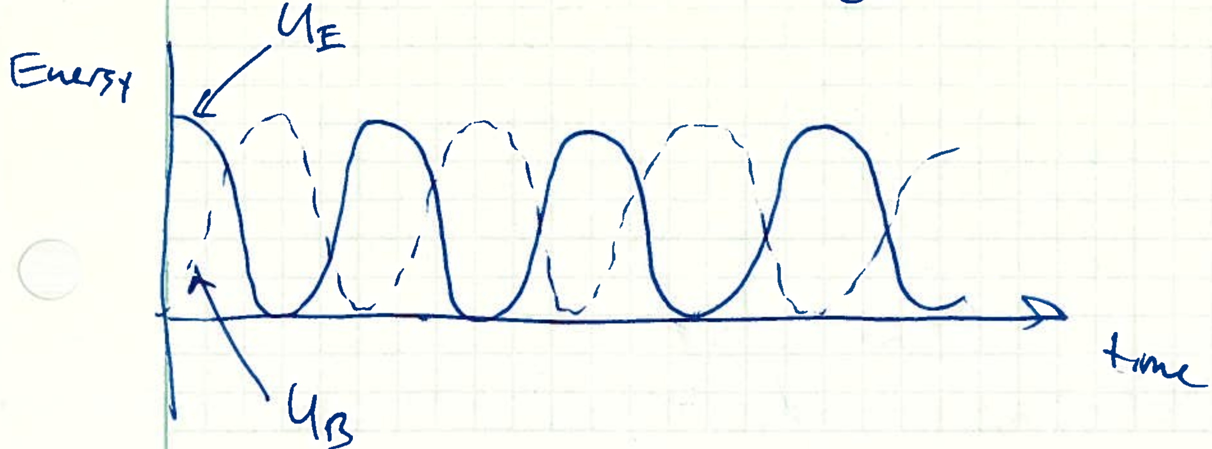
$$= \frac{1}{2} q_0^2 \underbrace{\omega_0^2 L}\, \sin^2(\omega_0 t + \delta)$$

$$\omega_0^2 = \frac{1}{LC} \quad \text{so} \quad \omega_0^2 L = \frac{1}{C}$$

$$= \frac{q_0^2}{2C} \sin^2(\omega_0 t + \delta)$$

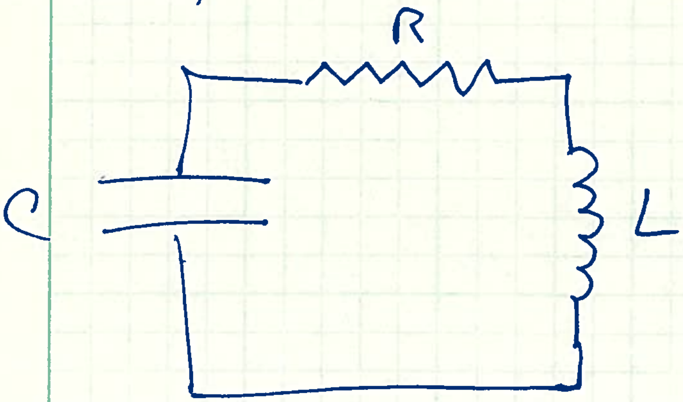
$$\text{Total energy} = U_E + U_B = \frac{q_0^2}{2C}\underbrace{\left[\cos^2(\omega_0 t + \delta) + \sin^2(\omega_0 t + \delta)\right]}_{1}$$

$$= \frac{q_0^2}{2C} = \text{constant}$$



LC oscillator with damping — RLC circuit

Add a resistor to the circuit: Electrical energy will be converted to heat in the resistor



Voltage Rule:

$$V_C + V_R + V_L = 0$$

$V_C \to \frac{q}{C}$

$V_R \to IR = \dot{q}R$

$V_L \to L\frac{dI}{dt} = L\frac{d^2q}{dt^2} = L\ddot{q}$

$$\therefore \frac{q}{C} + R\dot{q} + L\ddot{q} = 0$$

$$\boxed{\ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC}q = 0}$$

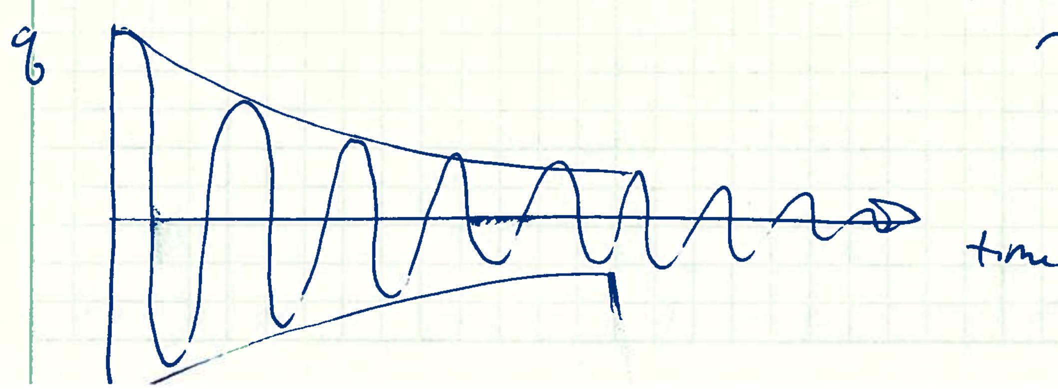
Simple Harmonic Oscillator with damping

Solution (light damping):

$$\boxed{q(t) = q_0 e^{-\gamma t/2} e^{i(\omega t+\delta)}}$$

where $\gamma = \frac{R}{L}$

and $\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$



Driven RLC circuit — Series

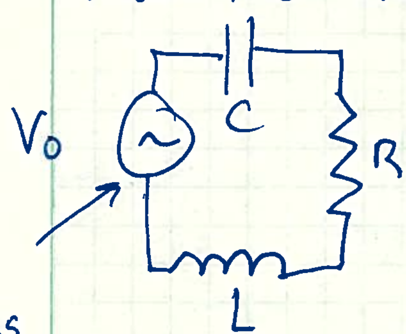
Suppose we have an oscillating circuit with a voltage source that varies in time as a cosine:

$$V_{source}(t) = V_0 \cos(\omega t) = V_0 e^{i\omega t}$$

forcing frequency

We choose δ (phase shift) $= \emptyset$ be choosing $t = \emptyset$ correctly.

Then the RLC series circuit looks like:



time varying voltage source

Voltage Loop Rule:

$$V_S = \cancel{V_R}\; V_C + V_R + V_L$$

$$\left(V_0 e^{i\omega t}\right) \quad \left(\frac{1}{C} q\right) \quad (R\dot{q}) \quad (L\ddot{q})$$

$$\boxed{\ddot{q} + \left(\frac{R}{L}\right)\dot{q} + \left(\frac{1}{LC}\right) q = \left(\frac{V_0}{L}\right) e^{i\omega t}}$$

Driven Harmonic Oscillator.

Steady State Solution:

$$q(t) = q_0 e^{i(\omega t + \delta)} \quad \text{or} \quad A e^{i(\omega t + \delta)}$$

where ~~$A = q_0$~~

$$q_0(\omega) = \frac{(V_0/L)}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}}$$

ω = driving frequency,

$\omega_0 = \frac{1}{\sqrt{LC}}$,

$\gamma = (R/L)$

and $$\delta(\omega) = -\tan^{-1}\left[\frac{\omega\gamma}{(\omega_0^2 - \omega^2)}\right]$$

Just like the forced mechanical oscillator, we have a resonance when $\omega \approx \omega_0$.

When $\omega \approx \omega_0$, the following things become very large:

1) the peak charge on the capacitor
2) the peak current in the circuit
3) the total energy stored in the C & L.