We've studied:

- Simple Harmonic oscillator.

Properties:

a) Goes at its "natural frequency": $\omega_0 = \sqrt{k/m}$

b) ~~Amplitude~~ Frequency is independent of amplitude.

c) Amplitude and phase are determined by initial conditions.

- Forced oscillators with no damping: $F(t) = F_0 e^{i\omega t}$

Properties:

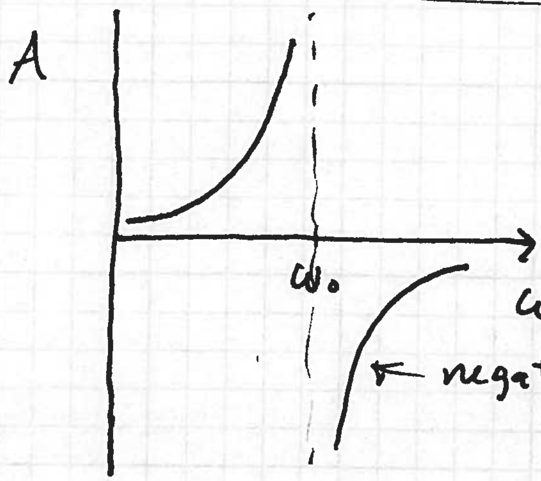
a) Oscillator goes at the forcing frequency.

b) Amplitude depends on how close the forcing frequency is to the natural frequency. Amplitude becomes very large if the forcing frequency is very close to ω_0.

c) ~~Phase~~ If $\omega < \omega_0$, oscillator is in phase with the external force $\Rightarrow$ no ~~phase shift~~ phase shift. If $\omega > \omega_0$, Oscillator ~~has~~ is 100% out-of-phase with the driving force: phase shift = $180° = \pi$ radians

Forced Oscillator, no damping: $x(t) = Ae^{i(\omega t+\delta)}$

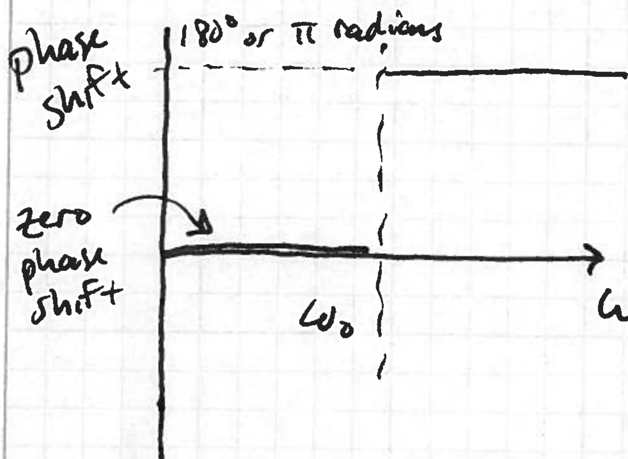
$\delta = \phi$, but A may be (+) or (−)



$$A = \frac{F_0}{m(\omega_0^2-\omega^2)}$$

$\omega \leftarrow$ forcing frequency

$\leftarrow$ negative A means 180° phase shift.



$\omega \leftarrow$ forcing frequency

For the forced oscillator the initial condition determines transient behavior. We will study this later.

Now let's do:

Simple Harmonic oscillator with damping

Real mechanical oscillators always have some resistive force which is non-conservative. Resistive forces turn mechanical energy into heat, where it is usually lost. Resistive forces must be modeled empirically. A very simple model is

$$F_{resistive} = -bv$$

$\leftarrow$ velocity of oscillator

constant

Force acts opposite the velocity

resistive Forces

Real ~~oscillators~~ ~~may~~ may not be accurately modeled by such a simple force law, but this model allows the Eq. of Motion to be easily solved. Also, this model is often OK when v is not ~~to~~ too large.

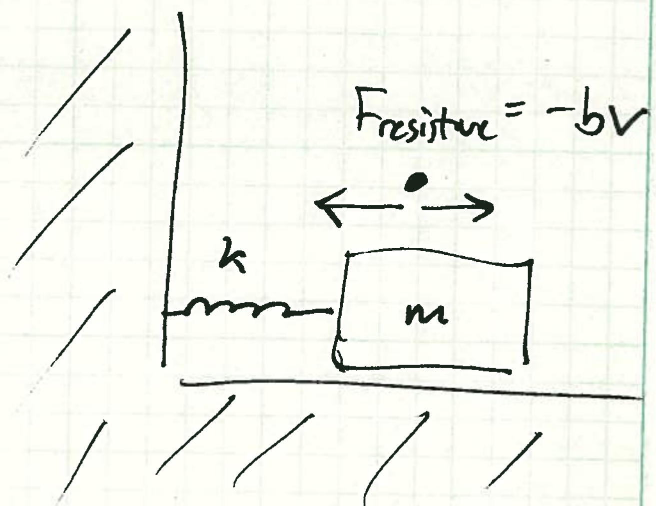
Eq. of Motion:

$\cancel{m\ddot{x}}$

$$-kx - bv = m\ddot{x}$$

↑

$v = \dot{x}$



$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \emptyset$$

↑

No driving force (for now)

Define $\gamma = \frac{b}{m}$. Then (Also $\omega_0^2 = k/m$)

$$\boxed{\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \emptyset}$$ Equation of Motion.

→ small γ means very little drag (resistive ~~resistance~~ force)

→ large γ means large drag.

Guessed Solution:

$$\boxed{x(t) = Ae^{i(\omega t + \delta)}}$$

⇒ A, δ, and ω are not yet known.

Substitute the guess:

$$\ddot{x} = -\omega^2 x \quad , \quad \dot{x} = i\omega x$$

Result: $-\omega^2 x + i\omega\gamma x + \omega_0^2 x = \phi$

x(t) divides out:

$$\boxed{-\omega^2 + i\omega\gamma + \omega_0^2 = \phi} \leftarrow$$ Purely Algebraic Equation for ω in terms of γ and ω_0.

Note that $\boxed{\omega = \text{purely real}}$ cannot satisfy this equation. ~~So so~~ Nor can ω = purely imaginary. So we must all ω to be complex, with both real and imaginary parts:

Let $\omega \equiv \omega_r + i\omega_i$, where ω_r = real and ω_i = ~~imaginary~~ real

Then our equation for ω says

$$-(\omega_r + i\omega_i)^2 + i(\omega_r + i\omega_i)\gamma + \omega_0^2 = \phi$$

Two equations:

$$-\omega_r^2 + \omega_i^2 - \omega_i\gamma + \omega_0^2 = \phi \qquad (1)$$

and $$i(-2\omega_r\omega_i + \omega_r\gamma) = \phi \qquad (2)$$

From (2) we have $\boxed{\omega_i = \frac{\gamma}{2}}$.

Then substitute into (1):

$$\boxed{\omega_r^2 = \omega_0^2 - \frac{\gamma^2}{4}}$$

Therefore $\omega = \left(\omega_0^2 - \frac{\gamma^2}{4}\right) + i\left(\frac{\gamma}{2}\right)$.

Then our guessed solution says

$$x(t) = Ae^{i(\omega t+\delta)}$$
$$= Ae^{i((\omega_r + i\omega_i)t+\delta)}$$
$$= Ae^{-\omega_i t} e^{i(\omega_r t+\delta)}$$

$$\boxed{x(t) = Ae^{-\gamma t/2} e^{i(\omega_r t+\delta)}} \quad \text{where } \omega_r = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

To simplify notation, let's ~~just call~~ change the name of ω_r to just ω:

$$\boxed{\begin{gathered} x(t) = Ae^{-\gamma t/2} e^{i(\omega t+\delta)} \\ \text{where } \omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \end{gathered}}$$

We still have 2 unknown constants: A & δ. These are determined by the initial conditions, just like the simple harmonic oscillator with no damping.

Comments:

① If we set $\gamma = \emptyset$, we recover the SHO solution

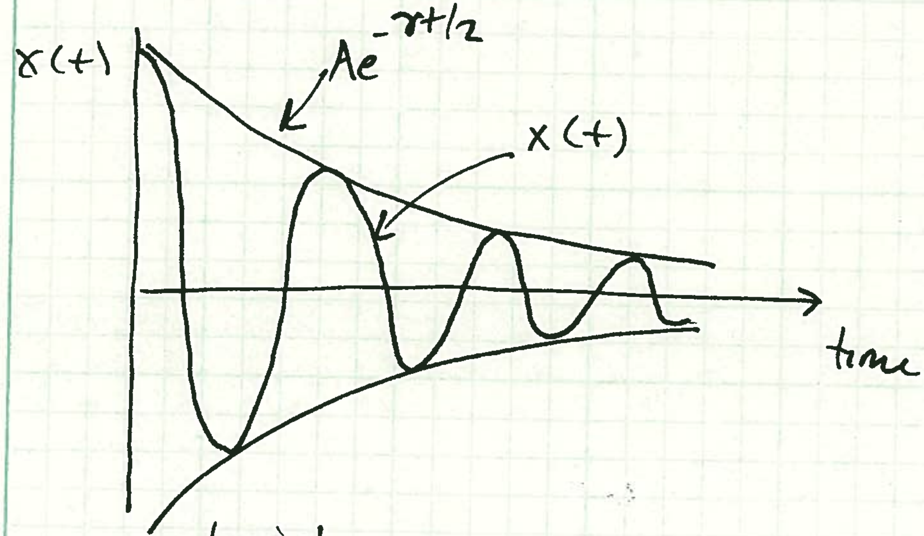
$$x(t) = Ae^{i(\omega_0 t+\delta)}$$

② For non-zero γ, we have an oscillation at frequency $\omega = \underbrace{\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}}$

↑ not the natural frequency!

But if γ is small (small damping), then the frequency of oscillation is very close to ω_0.

(3) The amplitude of the oscillation decays exponentially in time:



The mechanical energy of the oscillator is being converted into heat by the drag force.

Mathematically, the exponential decay shows up when we realized that ω must be complex in order to satisfy the equation of motion.