Why you should hate sine and cosine.

Sine and Cosine are not a good way of representing an oscillator or a wave.

Reason #1 They don't obey the normal rules of algebra.

Ex: $\frac{\cos\theta}{s} \stackrel{?}{=} co\theta$?!? [of course not]

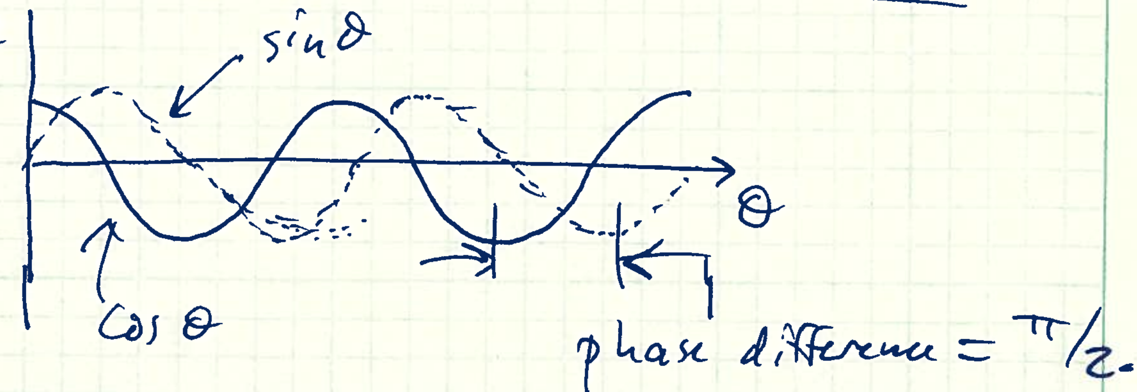
↑ s does not divide out!

Since they are not normal algebraic operations, we have to ~~~~ refer to long tables of trigonometric identities to use them. Trig identities are excellent opportunities to make mistakes. The table of trig identities in Hirose & Lonngren contains several errors! (Appendix B) (of Intro to Wave Phenomena)

Reason #2 Sine & Cosine obscure the true Amplitude and Phase, of an oscillation. and Frequency

Ex#1 $f(\theta) = \cos\theta$, $g(\theta) = \sin\theta$ } what is the phase difference between $f(\theta)$ & $g(\theta)$?

It appears to be zero, because the argument for both is just θ. But sine & cosine have a built-in phase difference of $\underline{\pi/2}$:



<u>Ex #2</u> What's the phase difference between
$$f(\theta) = \cos(\theta - \pi/2)$$
$$g(\theta) = \sin(\theta)$$
?

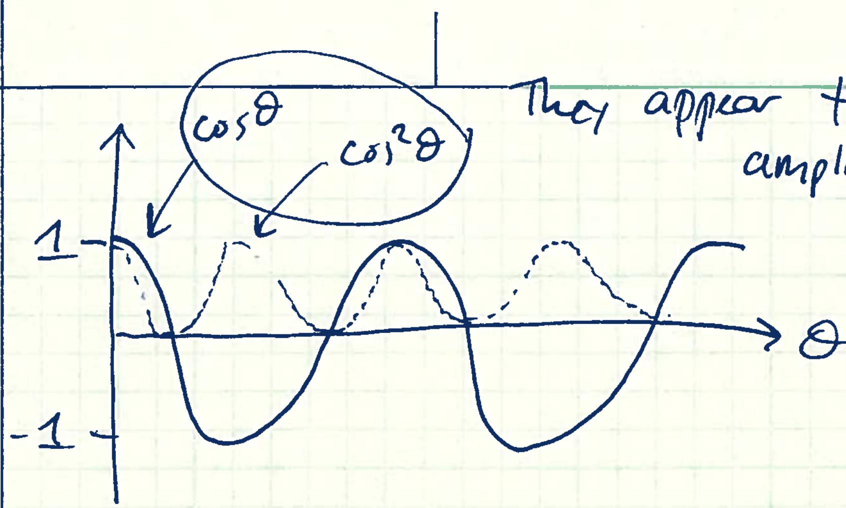
<u>Answer:</u> It <u>appears</u> to be $-\pi/2$, but it's really zero.

<u>Ex #3</u> What's the <u>amplitude</u> of $A\cos(\omega t + \delta)$?

<u>Answer:</u> It depends what you mean by amplitude. The full peak-to-peak variation is $\underline{2A}$. In this sense, A is really the half-amplitude.

<u>Ex #4</u> What's the amplitude of
$$[A\cos(\omega t + \delta)]^2 \,?$$

Answer: ~~by~~ after squaring the function, now the full peak-to-peak amplitude is simply A^2, <u>not</u> $(2A)^2$.



They appear to have the same amplitude, but $\cos\theta$ has a full ~~[illegible]~~ peak-to-peak variation twice as large as $\cos^2\theta$.

Ex #5 Does $\cos(\omega t)$ have the same frequency as $\cos^2(\omega t)$?

Answer: No $\rightarrow$ $\cos^2(\omega t)$ has twice the frequency as $\cos(\omega t)$.

$$\cos^2\omega t = \tfrac{1}{2}\left[1+\cos(2\omega t)\right]$$ $\leftarrow$ we have to use a trig identity to see that $\cos^2(\omega t)$ has twice the frequency.

~~Ex #6 Does $\cos^4(\omega t)$ have the same~~

$\Longrightarrow$ An oscillation is nothing but an amplitude, a phase, and a frequency, and sine & cosine are misleading about all three.

Reason #3 (why you should hate sine & cosine)

You might try to simplify your life by only using sine or only using cosine, but as soon as you differentiate, sin turns into cosine, and vice-versa. So you are (more or less) forced to use both. Since the equations of motion of physics are ODEs, we can't avoid taking derivatives.

The fundamental problem with sine & cosine are that they are designed to work well in geometry. But there is nothing particularly geometrical about a mass bouncing up and down on a spring. Sine and Cosine correctly describe a mass on a spring, but they are not the most natural way or simplest way to describe it. They are intended for geometry, not dynamics.

Simple Harmonic Oscillator again.

Equation of Motion is $\boxed{\ddot{x} + \omega_0^2 x = \emptyset}$

This equation demands a function $x(t)$ which is proportional to its own second derivative, (with a minus sign) Sine and Cosine happen to satisfy this, but so do exponential functions:

$$x(t) = e^{\omega_0 t} \quad \leftarrow \text{guessed solution}$$

$$\ddot{x}(t) = \omega_0^2 e^{\omega_0 t} = \omega_0^2 x(t)$$

Try it: $\ddot{x} + \omega_0^2 x$ =

$$= \omega_0^2 e^{\omega_0 t} + \omega_0^2 e^{\omega_0 t} = 2\omega_0^2 e^{\omega_0 t} \quad ?!?$$

$$\neq \emptyset \;!!$$

We wanted a (−) sign here, but we didn't get it. We need to replace (ω_0) with $(i\omega_0)$ where $i = \sqrt{-1}$.

$$x(t) = e^{i\omega_0 t} \quad \leftarrow \text{better guess}$$

$$\ddot{x}(t) = -\omega_0^2 e^{i\omega_0 t}$$

$$\ddot{x} + \omega_0^2 x$$

$$= -\omega_0^2 e^{i\omega_0 t} + \omega_0^2 e^{i\omega_0 t} = \emptyset \quad \checkmark \text{ Yes.}$$

So $e^{i\omega_0 t}$ is a solution to the simple harmonic oscillater equation of motion. It's a much ~~better~~ simpler way to represent the solution than sine & cosine because ~~it obeys all the normal rules of algebra ⇒ (no more trig identities!)~~, ~~and because~~

1. it obeys the normal rules of algebra (no more trig identities!)
2. It makes the amplitude, phase, and frequency much more obvious, and
3. taking derivatives is easy. (no more switching between sine & cosine.)

But it does raise some questions:

a) What does it mean to exponentiate the square root of -1?

b) How can ~~[illegible]~~ a complex function (real and imaginary) represent the position of a mass on a spring, which must always be a <u>real number</u>?

Complex Numbers

Complex numbers can be thought of as a pair of numbers: $z = (x, y)$

Addition is defined by

$$z_1 + z_2 \equiv (x_1 + x_2, y_1 + y_2)$$

and multiplication is defined by

$$z_1 z_2 \equiv (x_1, y_1)(x_2, y_2)$$
$$\equiv (x_1 x_2 - y_1 y_2,\ y_1 x_2 + x_1 y_2)$$

The multiplication rule is complicated, but its easy to remember if you think of $i \equiv \sqrt{-1}$ and

$$z_1 = x_1 + i y_1$$
$$z_2 = x_2 + i y_2 .$$

Then the multiplication rule follows directly from the usual rules of algebra:

$$z_1 z_2 = (x_1 + i y_1)(x_2 + i y_2)$$
$$= x_1 x_2 + i x_1 y_2 + i y_1 x_2 - y_1 y_2$$
$$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

So we don't usually use the (x, y) notation, instead we use the $x + iy$ notation.

Complex conjugation is defined such that it reverses the sign of the imaginary component:

$$z^* = (x + iy)^* \equiv x - iy$$

sometimes like to multiply a complex number its own complex conjugate. The result is

$$\begin{aligned} zz^* &= (x+iy)(x-iy) \\ &= (x^2+y^2) + i(xy-yx) \\ &= x^2+y^2 \end{aligned}$$

Example: if $z = 5+i3$, then $z^* = 5-i3$, and $zz^* = 25+9 = 34$.

Division: If a complex number appears in the denominator, the trick is the multiply numerator and denominator by the complex conjugate, to make the denominator purely real:

$$\frac{1}{x+iy} = \frac{(x-iy)}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2}.$$

Question: Can we define a cosine function for a complex number?

$$\cos(z) = ???$$

Answer: Yes, use the Taylor Series definition of cosine:

For real numbers, $\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \ldots$

So let's define

$$\cos(z) \equiv 1 - \frac{1}{2}z^2 + \frac{1}{4!}z^4 + \ldots$$

Note that $z^2 = zz$, not zz^*.

Example: $\cos(i) = 1 - \frac{1}{2}(i)^2 + \frac{1}{4!}(i)^4 + \ldots$

$$\cos(i) = \underbrace{1 + \frac{1}{2} + \frac{1}{4!} + \ldots}$$

$\cos(i)$ is a purely real number!

How about exponentials? Can we exponentiate a complex number?

Answer: Yes, again just extend the meaning of the exponential function by relying upon its Taylor Series:

real numbers: $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots$

So we define

$$e^z \equiv 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \ldots$$

Here is the magical thing: What happens if we exponentiate a purely imaginary number?

Let $z = iy$ ← no real part, pure imaginary

Then

$$e^z = e^{iy} = 1 + iy + \frac{1}{2!}(iy)^2 + \frac{1}{3!}(iy)^3 + \ldots$$

$$= \underbrace{\left(1 - \frac{1}{2}y^2 + \frac{1}{4!}y^4 + \ldots\right)}_{\cos(y)} + i\underbrace{\left(y - \frac{1}{3!}y^3 + \frac{1}{5!}y^5 + \ldots\right)}_{\sin(y)}$$

$$\therefore\ e^{iy} = \cos(y) + i\sin(y).$$

~~Sine~~ Sine & Cosine are based in geometry. But here they have appeared in a purely algebraic context. This shows that there is a deep connection between algebra and geometry.

Since we usually let θ represent the argument of sine and cosine functions, the last equation is usually written

$$\boxed{e^{i\theta} = \cos(\theta) + i\sin(\theta)}$$

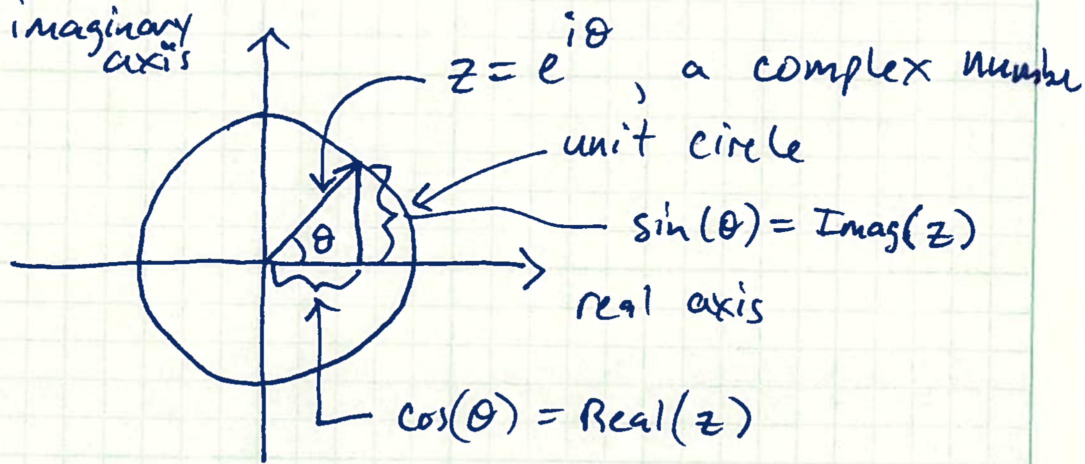
Euler's Formula — 1748

Feynman called this "one of the most remarkable, almost astounding formulas in all of mathematics".

Euler's Formula says that when you exponentiate a purely complex number, you get a number which has both real and imaginary components. These components ~~go up and down~~ oscillate forever as the argument increases, just like the x and y projections of the unit circle as the angle increases.

50 years after Euler, Wessel realized that Euler's Formula allows us to think of complex numbers being vectors in the complex plane:

Wessel's Picture of complex numbers as vectors:

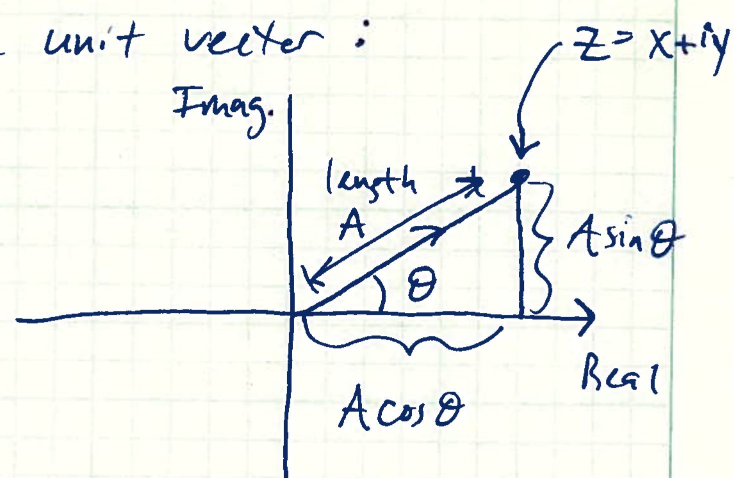


The magnitude of $e^{i\theta}$ is $\sqrt{\cos^2\theta + \sin^2\theta} = 1$

from geometry.

$\boxed{\begin{array}{l} e^{i\theta} \text{ is a unit vector in the complex plane.} \\ \text{It has length } 1. \end{array}}$

Any complex number can be represented this way simply by scaling the unit vector:



Let $z = x + iy$.

Then choose

$$A = \sqrt{x^2 + y^2}$$

and $\theta = \tan^{-1}(y/x)$.

Then $z = Ae^{i\theta}$

$\boxed{\begin{array}{l} Ae^{i\theta} \text{ is a vector in the complex plane.} \\ \text{It has magnitude } A \text{ and it makes} \\ \text{an angle } \theta \text{ with the real axis.} \end{array}}$

So we can think of complex numbers as vectors, and we can write them in two notations:

1. Cartesian notation: $z = x + iy$ ← a vector

2. Polar notation: $z = Ae^{i\theta}$ ← the same vector.

To go back and forth between notations:

$$\boxed{\begin{aligned} A &= \sqrt{x^2+y^2} \\ \theta &= \tan^{-1}(y/x) \end{aligned}}$$

Cartesian to Polar

$$\boxed{\begin{aligned} x &= A\cos(\theta) \\ y &= A\sin(\theta) \end{aligned}}$$

Polar to Cartesian.

But the Polar form is the most useful because it makes the amplitude and phase completely obvious to the eye:

$$z = \underset{\substack{\uparrow \\ \text{amplitude}}}{A}\, e^{i\theta} \leftarrow \text{phase.}$$

Also, in the polar form, every symbol obeys the standard rules of algebra.

Example: Can we divide by e? Answer: Yes:

$$\frac{z}{e} = \frac{Ae^{i\theta}}{e} = (Ae^{i\theta})(e^{-1}) = Ae^{i\theta-1} \text{ or } \underbrace{\left(\frac{A}{e}\right)}e^{i\theta}$$

Complex numbers are most useful and powerful when used in <u>polar form</u>:

$$z = \underbrace{A e^{i\theta}}$$

amplitude → A, phase → θ ← Polar form

↑ A vector in the complex plane

Polar form has these characteristics:

(1) The exponential is <u>purely imaginary</u>: $e^{i\theta}$

(2) The amplitude is <u>positive and purely real</u>: A = real, positive number.

Once we have ~~the~~ z written in polar form, then we can simply read-off the amplitude and phase by inspection, without ambiguity or caveats.

$$\text{polar form} = (\text{amplitude})\, e^{i(\text{phase})}.$$

To manipulate a complex number into polar form, just use the usual rules of alegbra (no trig identities).

<u>EX1</u>: $z = e^{i\omega t - \alpha}$ ← Is this Polar form?

Answer: <u>No</u>, because the exponential has a real component. But we can manipulate it into polar form:

$$z = e^{i\omega t - \alpha} = e^{i\omega t}\, e^{-\alpha} = \underbrace{\left(e^{-\alpha}\right)}_{\text{purely real}} e^{i(\omega t)}$$

purely imaginary. ✓

Polar Form ✓

What's the amplitude? Answer: $e^{-\alpha}$.

What's the phase? Answer: ωt.

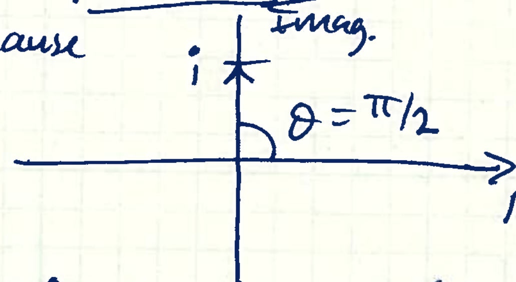
<u>Ex:2</u> $z = i e^{i\omega t}$ ← Is this polar form?

Answer: <u>No</u> because the (i) in front is <u>not</u> <u>purely real</u>.

Manipulate it: ~~the~~ $z = \underbrace{i}\, e^{i\omega t}$ ← First convert (i) to polar form

$$\boxed{i = e^{i\pi/2}}$$

because



so $z = (i)(e^{i\omega t}) = (e^{i\pi/2})(e^{i\omega t}) = e^{i(\omega t + \pi/2)}$

purely imaginary

Polar Form ✓

purely real

What's the amplitude? Answer: 1.

What's the phase? Answer: $\omega t + \pi/2$.

⟹ Multiplication by (i) represents a phase shift of $\pi/2$.

Ex 3: $z = -e^{i\omega t}$ ← Is this Polar Form?

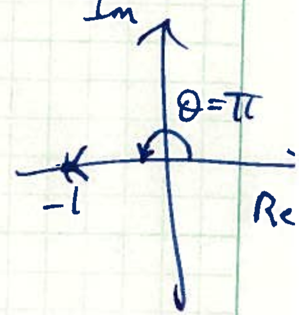
Answer: Not really, because the amplitude should be positive, not negative.

Manipulate it: $z = -e^{i\omega t} = \underbrace{(-1)}\left(e^{i\omega t}\right)$

$\uparrow\ -1 = e^{i\pi}$

$= e^{i\pi} e^{i\omega t}$

$= e^{i(\omega t + \pi)}$ Polar Form ✓



What's the amplitude?: Answer: 1

What's the phase?: Answer: $\omega t + \pi$.

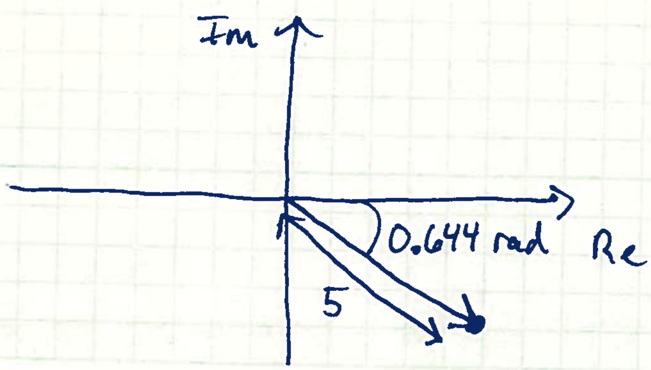
⇒ $\boxed{\text{Multiplication by } -1 \text{ represents a phase shift of } \pi.}$

Ex 4: $z = 4 - 3i$ ← Manipulate it into Polar Form.

$A = \sqrt{4^2 + (-3)^2} = 5$

$\theta = \tan^{-1}\left(-3/4\right) = -0.644$ radians

$\therefore\ z = 5e^{-i(0.644)}$



But how can we use complex functions to represent a physical quantity like position, which must be purely real?

Answer: We'll simply agree to take the real part of the answer at the very end of the calculation.

$\Rightarrow$ In fact, we usually <u>won't even bother to take the real part explicitly</u>, we'll just leave the solution in complex form and agree that we only care about the real part.

Ex: <u>Simple Harmonic Oscillator.</u>

Equation of Motion: $\ddot{x} + \omega_0^2 x = 0.$

Guessed Solution: $x(t) = Ae^{i(\omega_0 t + \delta)}$

Check it: $\dot{x}(t) = iA\omega_0 e^{i(\omega_0 t + \delta)}$

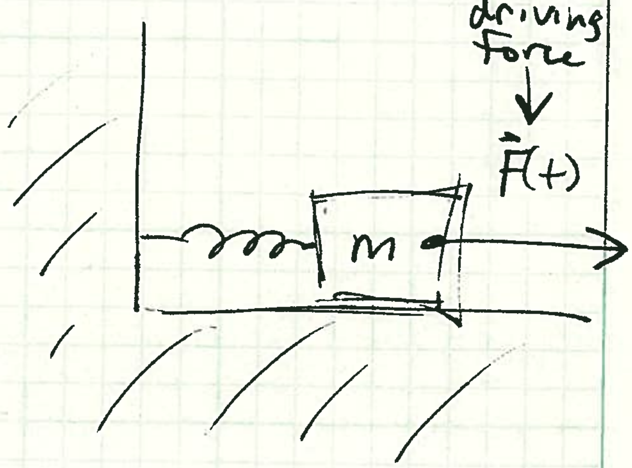
$$\ddot{x}(t) = -A\omega_0^2 e^{i(\omega_0 t + \delta)} = -\omega_0^2 x(t)$$

$$\Rightarrow -A\omega_0^2 e^{i(\omega_0 t + \delta)} + A(\omega_0^2) e^{i(\omega_0 t + \delta)} = 0 \checkmark$$

Final Answer: $\boxed{x(t) = Ae^{i(\omega_0 t + \delta)}}$

(Of course, we know that only the real part is <u>physically relevant</u>: $\text{Re}[x(t)] = A\cos(\omega_0 t + \delta)$, but we

Forced Oscillator



Imagine a mass on a spring which is subject to a second force that varies in time.
The second force could be due to a motor which is attached, for example.
The Equation of Motion is $F = ma$

$$-kx + F(t) = m\ddot{x}$$

Eq of Motion, forced oscillator $\boxed{\ddot{x} + \omega_0^2 x = \frac{F(t)}{m}}$, where $\omega_0 \equiv \sqrt{k/m}$

To be concrete, lets assume that the force has the form: $F(t) = F_0 \cos(\omega t)$, where ω is the frequency of the motor which drives the oscillator.

ω_0 = natural frequency of the oscillator, with no forcing
ω = frequency of the external driving force

Keep in mind that $\omega_0 \neq \omega$ ← we can drive at any frequency, independent of k and m.

We'd like to solve this with the mathematics of complex exponentials, but will it work?

Question Why are we allowed to use complex exponentials?

Answer: Let's imagine that we are solving a fictious problem where x and F are complex. Let

$$x \equiv x_r + i x_i$$

real part of x — imaginary part of x

and $F = F_r + i F_i$

real — ~~complex~~ imaginary

Then our fictious equation of motion is

$$\frac{d^2(x_r + i x_i)}{dt^2} + \omega_0^2 (x_r + i x_i) = \frac{F_r + i F_i}{m}$$

Gather together real & imaginary terms:

$$\left[\frac{d^2 x_r}{dt^2} + \omega_0^2 x_r - \frac{F_r}{m}\right] + i\left[\frac{d^2 x_i}{dt^2} + \omega_0^2 x_i - \frac{F_i}{m}\right] = 0$$

For this to be true for all times, both the real & imaginary parts should be zero.

$$\boxed{\begin{aligned} &\frac{d^2x_r}{dt^2} + \omega_0^2 x_r - \frac{F_r}{m} = \emptyset \\ \text{and} \quad &\frac{d^2x_i}{dt^2} + \omega_0^2 x_i - \frac{F_i}{m} = \emptyset \end{aligned}}$$

or

$$\boxed{\begin{aligned} &\ddot{x}_r + \omega_0^2 x_r = \frac{F_r}{m} \\ \text{and} \quad &\underbrace{\ddot{x}_i + \omega_0^2 x_i = \frac{F_i}{m}} \end{aligned}}$$

This is the true physics equation of motion

This is the same equation.

So by solving the fictitious complex equation, we are effectively solving the true physics equation, as the real (or imaginary) component

So let's assume a driving force of

$$F(t) = F_0 e^{i\omega t} \quad \text{which really means} \quad F(t) = F_0 \cos(\omega t)$$

And let's pretend that x is a complex variable

Now we solve:

$$\ddot{x} + \omega_0^2 x = \frac{F_0}{m} e^{i\omega t}$$

Guessed solution: Let's try the same solution that we had for the simple harmonic oscillator:

Guessed Solution: $x(t) = Ae^{i(\omega_0 t+\delta)}$ which really means

$$x(t) = A\cos(\omega_0 t+\delta)$$

here we guess that the oscillator still goes at its natural frequency (ω_0). Is this correct? Try it:

$$\ddot{x} = -\omega_0^2 x$$

Therefore $-\omega_0^2 x + \omega_0^2 x \overset{?}{=} \frac{F_0}{m} e^{i\omega t}$

$$\phi \overset{?}{=} \frac{F_0}{m} e^{i\omega t}$$

$F_0 = \phi$!! ← This solution only works if the driving force is zero.

So we need a better guess. Lets assume that the oscillator goes at the driving force frequency (ω) rather than the natural frequency:

Guessed Solution: $x(t) = Ae^{i(\omega t+\delta)}$

phase, real.

Amplitude, real

forcing frequency not natural frequency.

Try it: $\ddot{x} = -\omega^2 x = -\omega^2\left(Ae^{i(\omega t+\delta)}\right)$

Therefore we have

$$-\omega^2\left(Ae^{i(\omega t+\delta)}\right) + \omega_0^2\left(Ae^{i(\omega t+\delta)}\right) = \frac{F_0}{m}e^{i(\omega t)}$$

All the $e^{i\omega t}$ factors cancel:

$$Ae^{i\delta}\left(+\omega_0^2 - \omega^2\right) = \frac{F_0}{m}$$

$$A = \frac{F_0 e^{-i\delta}}{m(\omega_0^2-\omega^2)}$$

The LHS is <u>real</u> because the amplitude is always real for a polar form complex number

Therefore the RHS must be real:

$$e^{-i\delta} = \text{real number (because } F_0, m, \omega_0^2 \text{ \& } \omega^2 \text{ are real).}$$

$$= \cos(\delta) - i\underbrace{\sin(\delta)}$$

$$\sin(\delta) = 0 \Rightarrow \boxed{\delta = 0}$$

Phase shift is zero $\Rightarrow$ oscillator moves in phase with the driving force. (But see caveat below.)

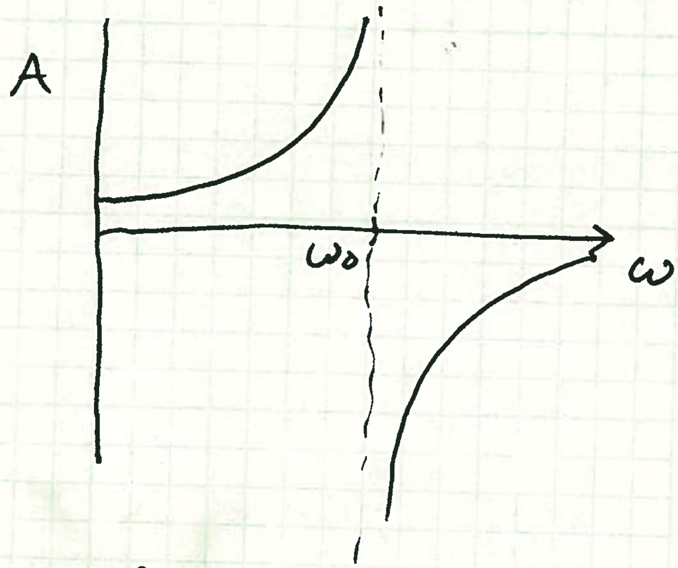
So our result is

$$\boxed{A = \frac{F_0}{m(\omega_0^2-\omega^2)}} \leftarrow$$

Our guessed solution works, but the amplitude is no longer a free parameter

The amplitude is determined by how close the driving frequency is to the natural frequency:

Amplitude of Motion vs. Driving Frequency



Comments:

1. If driving frequency is small compared to natural frequency, then A is positive and small $\Rightarrow$ Small oscillation,
   $\Rightarrow$ Phase difference between oscillator and driving force is zero.

2. If driving frequency is large compared to the natural frequency, then A is negative and small $\Rightarrow$ small oscillation
   $\Rightarrow$ 180° out of phase with driving frequency

3. If the driving frequency is similar to the natural frequency, then the amplitude is very large.