Dispersion

We have been studying systems which are governed by the classical wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v_p^2}\frac{\partial^2 y}{\partial t^2}$$

v_p is the "phase velocity", which is the speed at which a ~~point~~ peak or trough in the wave will travel.

The normal modes are the harmonic solutions to this equation. They are

$$y_n(x) = Ae^{ikx}$$

(recall that when there are no boundary conditions, k is continuous)

with associated frequency

$$\omega = k v_p$$

~~We can~~ Then the time development of the normal mode is

$$y_n(x,t) = \left(Ae^{ikx}\right)e^{i\omega t} = Ae^{i(kx+\omega t)}$$

We can explicitly confirm that this satisfies the equation of motion:

$$\frac{\partial^2 y}{\partial x^2} = -k^2 Ae^{i(kx+\omega t)}$$

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 Ae^{i(kx+\omega t)}$$

Substitute into Eq. of motion:

$$-k^2 A e^{i(kx+\omega t)} \stackrel{?}{=} \frac{1}{v_p^2}\left(-\omega^2 A e^{i(kx+\omega t)}\right)$$

$$-k^2 \stackrel{?}{=} -\frac{\omega^2}{v_p^2}$$

$$v_p = \frac{\omega}{k}.$$

We see that the normal mode $Ae^{i(kx+\omega t)}$ is a solution, as long as the phase velocity is ω/k. We can rewrite this condition.

$$\boxed{\omega = v_p k}$$

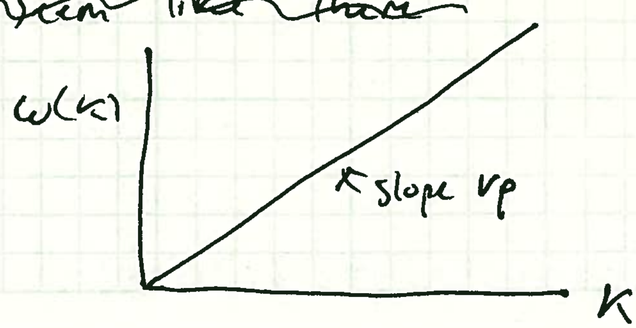
This equation describes all the physics of this system. We can think of this equation as telling us how the normal mode frequency (ω) depends on the normal mode wave number (k):

$$\omega(k) = \underbrace{v_p k}$$

linear dependence.

normal mode frequency depends on the wave number.

~~It may seem like there~~

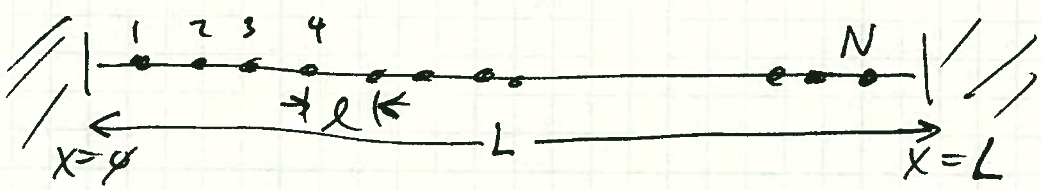


It may seem like there is no other possible relationship between ω and k, but this particular linear relationship only holds true for the classical wave equation. If any other equation of motion is used, then in general there will be some other relationship between ω and k.

An example is the <u>loaded string</u>. For that system, the normal mode frequencies are

$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right)$$

We can re-write this as an $\omega(k)$ function:



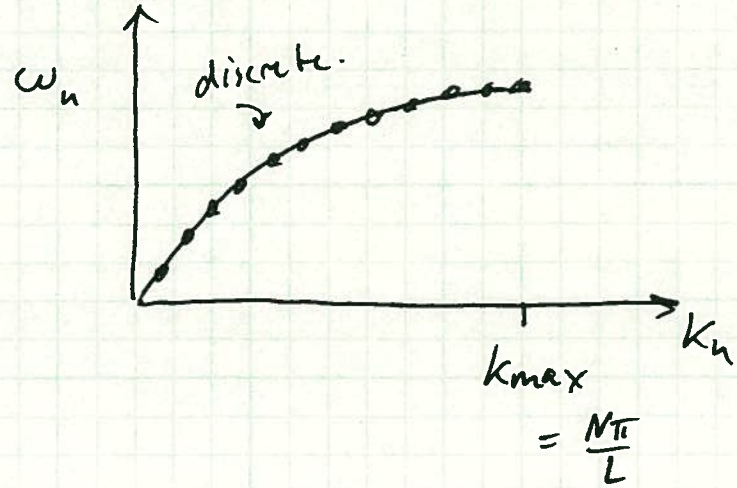
$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi\, l}{\underbrace{2(N+1)l}_{L}}\right) = 2\omega_0 \sin\left(\frac{n\pi l}{2L}\right)$$

~~now~~ $k_n = \frac{n\pi}{L}$, so we can re-write as

$$\boxed{\omega(k_n) = 2\omega_0 \sin\left(\frac{k_n l}{2}\right)}$$

where n goes from 1 to N.

This is a non-linear relationship between k_n and ω_n



The reason why this system does not have a linear relationship between ω & k is because its equation of motion is not the simple classical wave equation. It's equation is

$$\ddot{y}_p + 2\omega_0^2 y_p - \omega_0^2 (y_{p+1} + y_{p-1}) = \emptyset$$

In general, the relationship between ω & k is determined by the equation of motion, which is determined by the physics of the system.

Notice that when $N \to \infty$, the loaded string becomes a continuous string. This is equivalent to zooming-in on the linear part of the equation near $k_n = \emptyset$.

For any ~~ste~~ system, the relationship between ω and k is called the "dispersion relation"

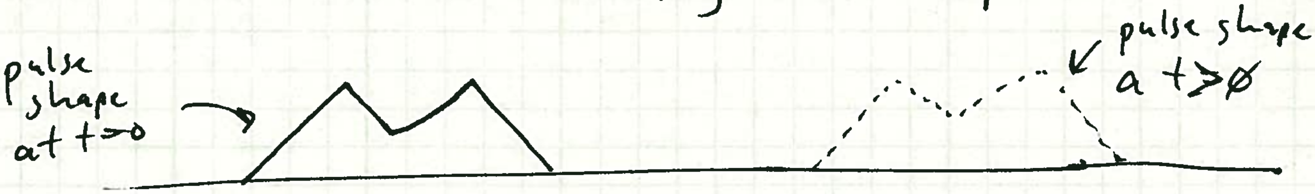
The classical wave equation has a linear dispersion relation:

$$\boxed{\omega(k) = v_p k}$$ ← linear dispersion relation (classical wave equation)

whereas the loaded string has a non-linear dispersion relation:

$$\boxed{\omega(k_n) = 2\omega_0 \sin\left(\frac{k_n l}{2}\right)}$$ ← non-linear dispersion relation (loaded string)

A system which has a linear dispersion relation has a special property: A propagating pulse will travel without changing its shape:



We can show this as follows. The pulse can be described as a sum over normal modes. But the normal modes are continuous, so the sum over normal modes is a Fourier Transform:

$$y(x,t) = \left[\frac{1}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} dk}_{\text{sum over}} \underbrace{A(k)}_{\text{its coefficient}} \underbrace{e^{i(kx)}}_{\text{normal mode}} \underbrace{e^{-i\omega t}}_{\text{its frequency}}\right]$$

$$y(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \; A(k) \, e^{i(kx-\omega t)}$$

Now suppose that the system has a <u>linear dispersion</u> relation:

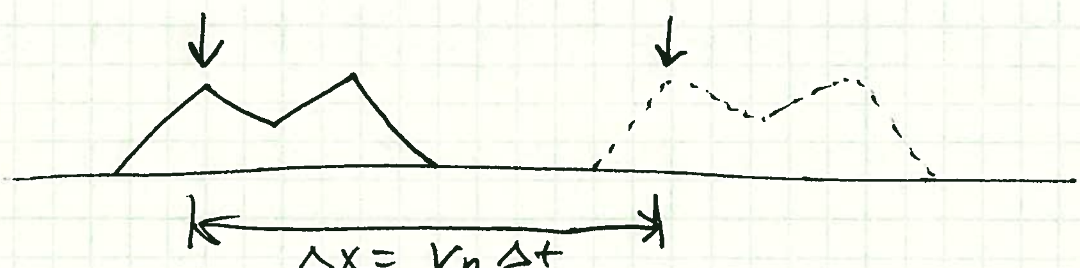
$$\omega = v_p k$$

Then we have

$$y(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \; A(k) \, e^{i(kx - (v_p k)t)}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \; A(k) \, e^{ik\underbrace{(x - v_p t)}}$$

This says that as time goes forward, if we keep advancing x at speed v_p, then the value of y will stay the same. So the shape of the pulse does not change



Therefore, if $\omega = v_p k$ for the system, then pulses <u>do not disperse</u>. They <u>maintain</u> <u>their shape</u>.

> A linear dispersion relation means that pulses do not disperse.

This is a special case behavior for systems with linear dispersion relations. But suppose that we have a non-linear dispersion relation. For example, suppose

$$\omega \sim k^2 .$$

This happens in quantum mechanics when describing a free particle. In that case,

$$\omega = \frac{\hbar k^2}{2m}.$$

How does a pulse propagate in a system like this?

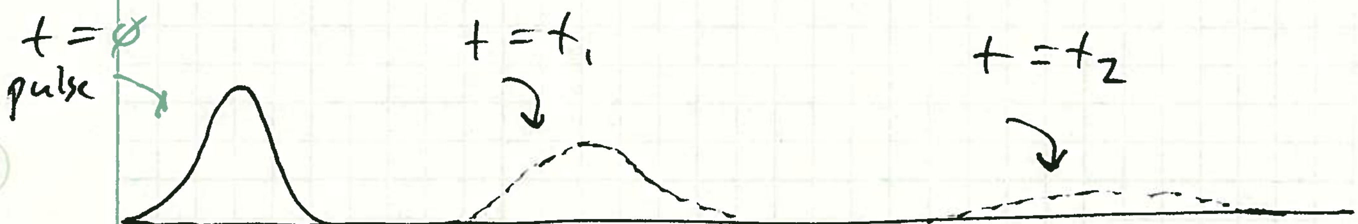
$$y(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, A(k) e^{ikx} e^{-i\omega t}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk A(k) \, e^{ikx} e^{-i\left(\frac{\hbar k^2}{2m}\right)t}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk A(k) \, e^{ik\left(x - \frac{\hbar k}{2m} t\right)}$$

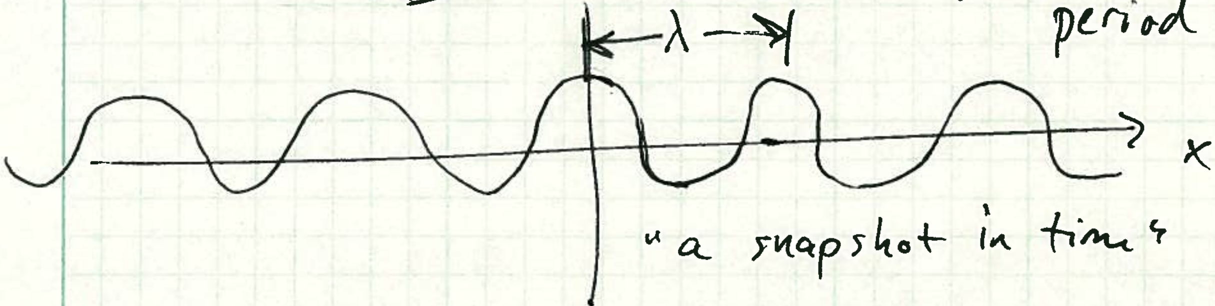
Here, as time goes forward, we need to advance x at a speed of $v_p = \frac{\hbar k}{2m}$ to keep the argument of the exponential the same.

But the speed $v_p = \frac{\hbar k}{2m}$ is different for every ~~normal~~ wavenumber (k). That is, each normal mode ~~as~~ advances at its own velocity, they do not advance together. This means that the various normal modes will disperse; with some travelling fast, and some travelling slowly, and the pulse will dissappear.



Further thoughts on Dispersion

Consider a <u>perfect</u> traveling wave; wavelength λ, period T.



At one location, say $x=0$, as time goes forward.

The wavelength is $\underline{\lambda}$, and the period is $\underline{T}$.

By definition of λ and T, this wave advances at a speed of

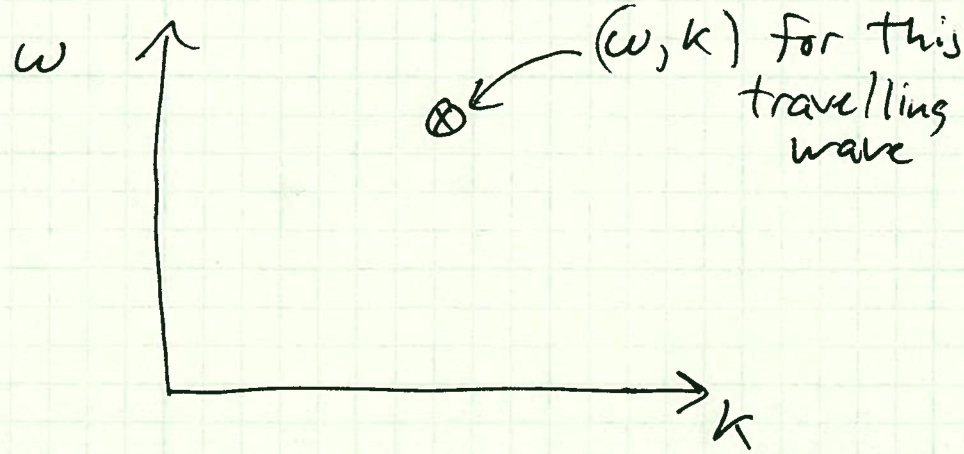
$$V = \frac{\lambda}{T} = \frac{\left(\frac{2\pi}{k}\right)}{\left(\frac{2\pi}{\omega}\right)} = \frac{\omega}{k}$$

This is the <u>phase</u> <u>velocity</u>, <u>by definition</u>.

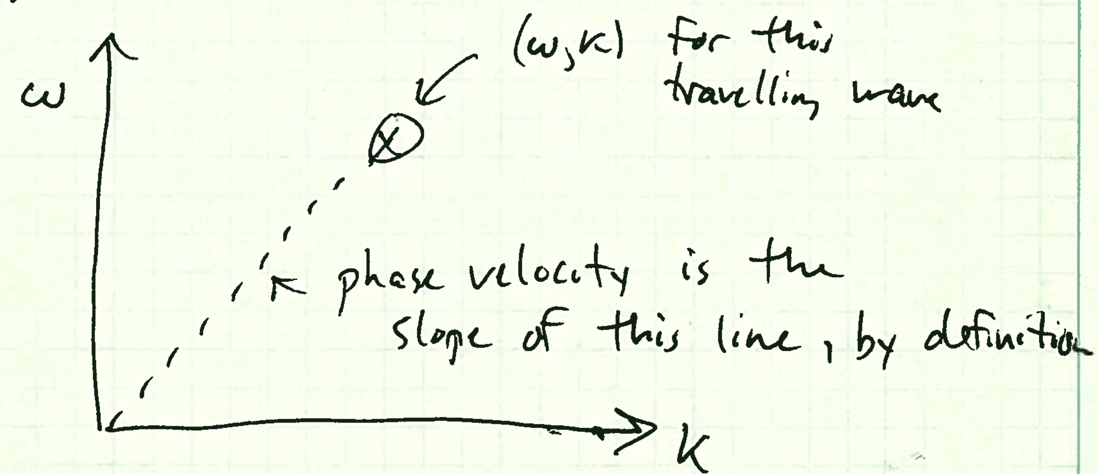
$$\boxed{V_{phase} \equiv \frac{\omega}{k} \quad \text{by definition}}$$

The phase velocity of a perfect travelling wave is the ratio of ω and k, by definition.

Lets plot ω & k for this travelling wave:



By definition, the ~~the~~ phase velocity ~~is the slope~~ of this wave is the slope ~~of~~ of a line from $(0,0)$, to (ω, k):



Now suppose our physical system is described by the classical wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \alpha \frac{\partial^2 y}{\partial t^2}$$

α ← some constant.

If the physical system is a string with tension (T) and mass density (ρ), then we know that

$$\alpha = \frac{\rho}{T}.$$

Or, if the physical system is electromagnetic waves in vacuum, then

$$\alpha = \mu_0 \epsilon_0$$

Let's guess a travelling wave solution to the classical wave equation:

$$y(x,t) = Ae^{i(kx-\omega t)}$$

What requirements does this place on k & ω?

Let's substitute:

$$\frac{\partial^2 y}{\partial x^2} = -k^2\left(Ae^{i(kx-\omega t)}\right)$$

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2\left(Ae^{i(kx-\omega t)}\right)$$

So the Classical Wave Equation says:

$$\left[-k^2\left(Ae^{i(kx-\omega t)}\right)\right] = \alpha\left[\left(-\omega^2\left(Ae^{i(kx-\omega t)}\right)\right)\right]$$

$$-k^2 = -\alpha\omega^2$$

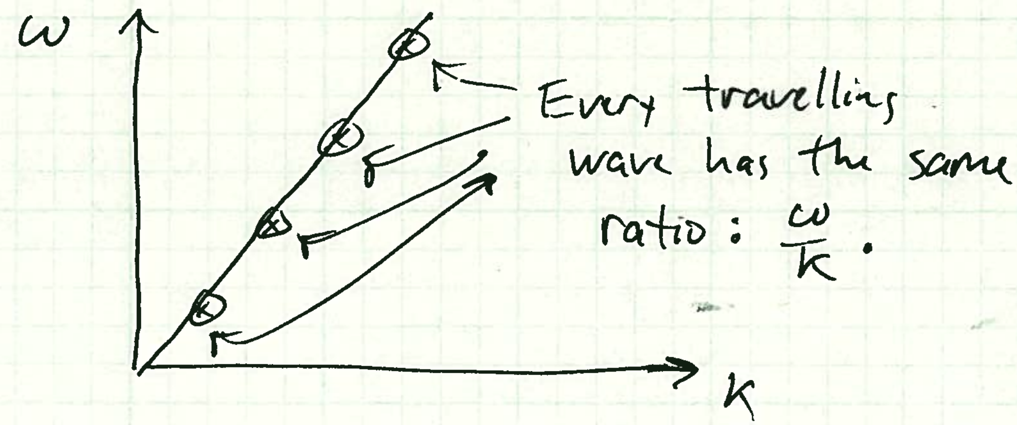
$$\omega^2 = \frac{1}{\alpha}k^2$$

or $$\frac{\omega}{k} = \frac{1}{\sqrt{\alpha}} = \text{constant}$$

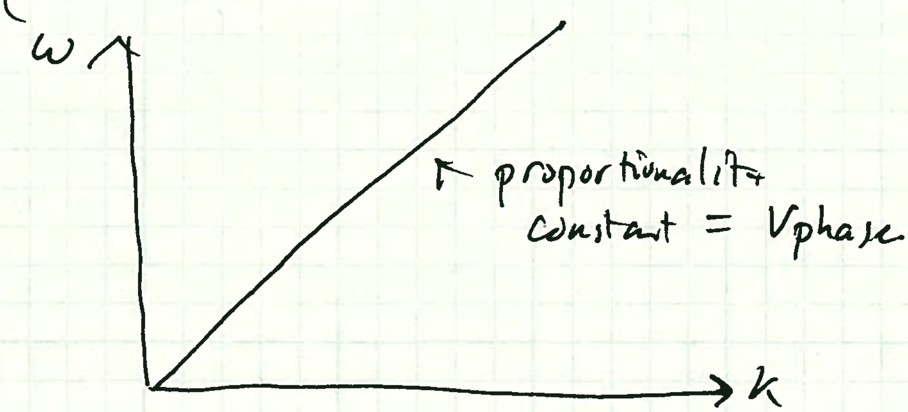
Now, by definition, $\frac{\omega}{k}$ = phase velocity = V_p.

So the classical wave equation requires that the phase velocity is a constant, independent of k, and independent of ω.

So if we plot ω & k for a system described by the classical wave equation, every valid travelling wave will fall on a straight line that passes through the origin:



So what the classical wave equation requires is that the frequency (ω) be directly proportional to (k) (the wave number).



Conversely, we could say that <u>any system that has direct proportionality</u> between ω & k is <u>described by the classical wave equation</u>

So let <u>β</u> be some <u>constant</u>. If we find ~~then~~ a system for which

$$\omega = \beta k,$$

then we can immediately conclude that <u>the equation of motion of</u> the system is <u>the classical wave equation</u>. Further, we can immediately infer that the phase velocity is

$$V_{phase} \equiv \frac{\omega}{k} \quad (\text{by definition})$$

$$V_{phase} = \frac{(\beta k)}{k} \leftarrow \text{for this particular system}$$

$$V_{phase} = \beta = \text{a constant}$$

But in general we should <u>not</u> expect that the system is described by the classical wave equation. Then ω is a more complicated function of k:

$$\omega(k) = \text{some complicated function of } (k).$$

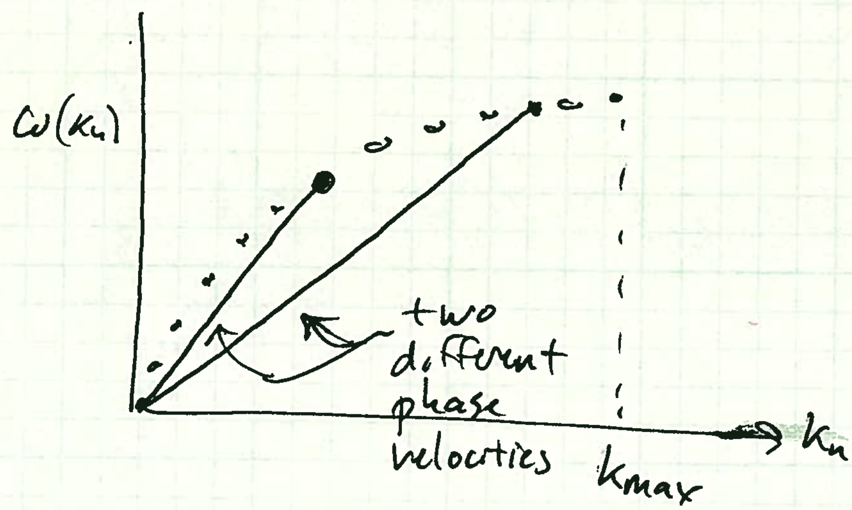
It will still be true that the <u>ratio</u> of ω & k is <u>still the phase velocity</u>, because this is true by definition.

$$V_{phase} \equiv \frac{\omega(k)}{k} \quad \text{by definition.}$$

So for a more complicated system, the ratio $\frac{\omega(k)}{k}$ will <u>not</u> be a constant.

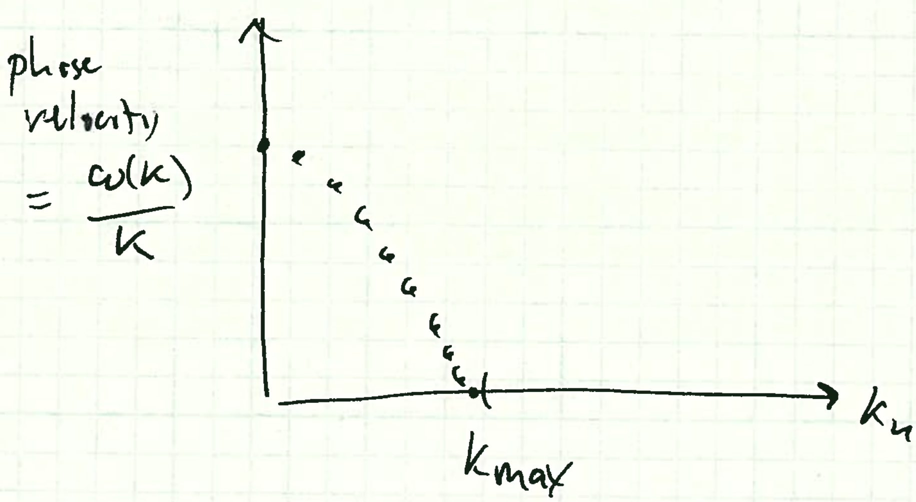
For example, for the loaded string,

$$\omega(k_n) = 2\omega_0 \sin\left(\frac{k_n l}{2}\right)$$ (k_n is discrete for this system)



For this system

$$V_{phase} = \frac{\omega(k_n)}{k_n} = 2\omega_0 \underbrace{\left(\frac{\sin\left(\frac{k_n l}{2}\right)}{k_n}\right)}_{\text{not a constant, depends upon } k_n.}$$



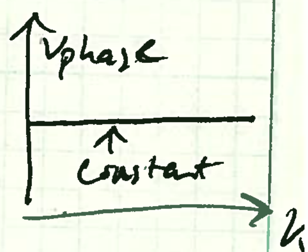
The $\omega(k)$ function is called the "dispersion relation", and the ratio of the dispersion relation and k is the phase velocity. The dispersion relation for the classical wave equation is

$$\omega(k) = (\text{some constant})\, k$$

and the phase velocity for the classical wave equation is

$$V_{phase} = \frac{\omega(k)}{k} = \frac{(\text{some constant})\, k}{k}$$

$$V_{phase} = \text{some constant}$$



So we can write

$$\omega(k) = V_{phase}\, k. \quad \text{for the classical wave equation}$$

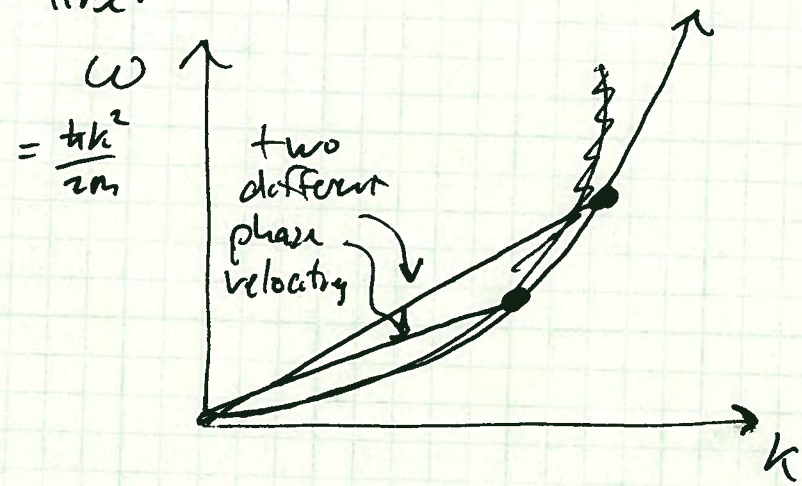
Another important case is free particles in quantum mechanics. These particles have the following dispersion relation:

$$\omega(k) = \frac{\hbar k^2}{2m} \leftarrow \text{quantum free particle}$$

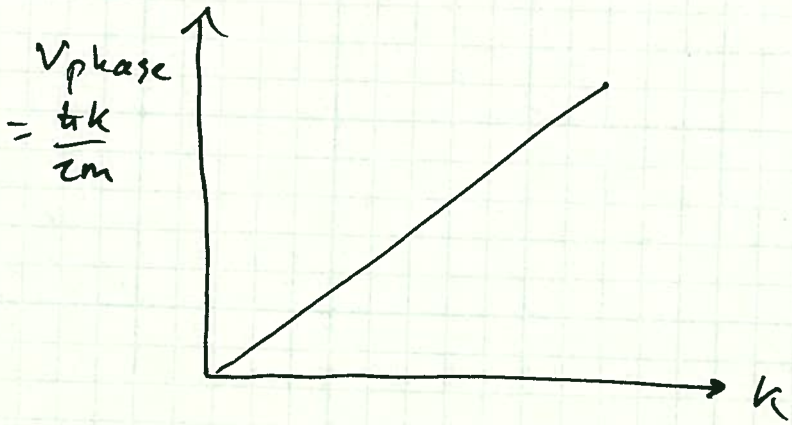
and their phase velocity is

$$V_{phase} = \frac{\omega(k)}{k} = \frac{\left(\frac{\hbar k^2}{2m}\right)}{k} = \underbrace{\frac{\hbar k}{2m}}_{\text{depends upon } k} = \text{not a constant}$$

It looks like:



The phase velocity increases linearly with k:



Now consider a <u>generic</u> system, described by some dispersion relation $\omega(k)$. <u>How would a pulse travel through a this system?</u>

<u>Answer</u>: the pulse is a <u>linear combination</u> of travelling waves, and each travelling wave propagates forever.

$$\text{pulse} = y(x,t) = \frac{1}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty}}_{\text{a continuous linear combination}} dk \; \underbrace{A(k)}_{\text{its coefficient}} \underbrace{e^{i(kx-\omega t)}}_{\text{travelling wave}}$$

For some $A(k)$ which describes the pulse in k-space

Now ~~the~~ $\omega = \omega(k)$, so

$$\text{pulse} = y(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, A(k)\, e^{ik\left(x - \frac{\omega(k)}{k} t\right)}$$

Now suppose the system ~~is~~ has a simple dispersion relation: $\omega(k) = (\text{some constant})\, k = v_{phase}\, k$

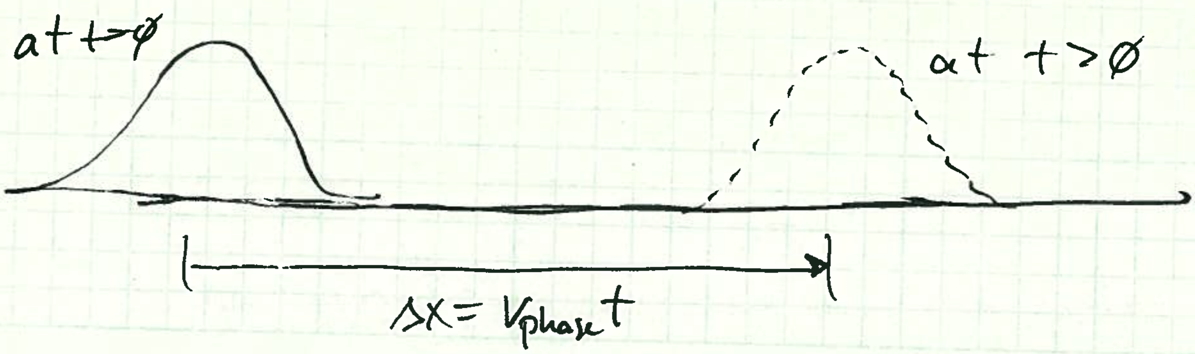
Then

$$y_{simple}(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, A(k)\, e^{ik(x - v_{phase} t)}$$

Or suppose that the system is a quantum free particle:

$$y_{quantum}(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, A(k)\, e^{ik\left(x - \left(\frac{\hbar k}{2m}\right) t\right)}$$

For ysimple, we can surf along at a constant value of y, if we advance x ~~[illegible]~~ at the constant rate of vphase. In other words, the entire pulse advances without changing its shape at a speed of vphase:

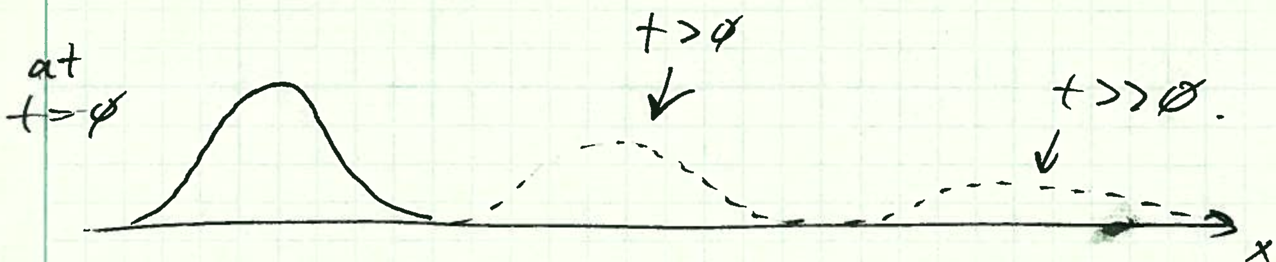


But what about the quantum free particle?
To keep the phase constant, we have to advance at a different rate for every wavenumber k which makes up the pulse.

$$\psi_{quantum} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, A(k)\, e^{\underbrace{ik\left(x - \frac{\hbar k}{2m} t\right)}}$$

there is no way to hold this constant for all k at the same time.

The problem is that all the component travelling waves which make up the pulse are travelling at different velocities. In other words, the are dispersing. And the pulse will dissappear as it goes forward in time:



So the simple dispersion relation really means "no dispersion"

$$\boxed{\begin{array}{l}\text{If } \omega(k) = (\text{some constant})\, k \text{, then} \\ \text{there will be no dispersion. Pulses will} \\ \text{travel forever with the same shape.}\end{array}}$$

Therefore the classical wave equation describes systems which have no dispersion. In these systems, a pulse can travel forever. An example of this in nature is electromagnetic waves in vacuum, (or waves on an ideal string.)

Information ~~transmit~~ transmission and group velocity

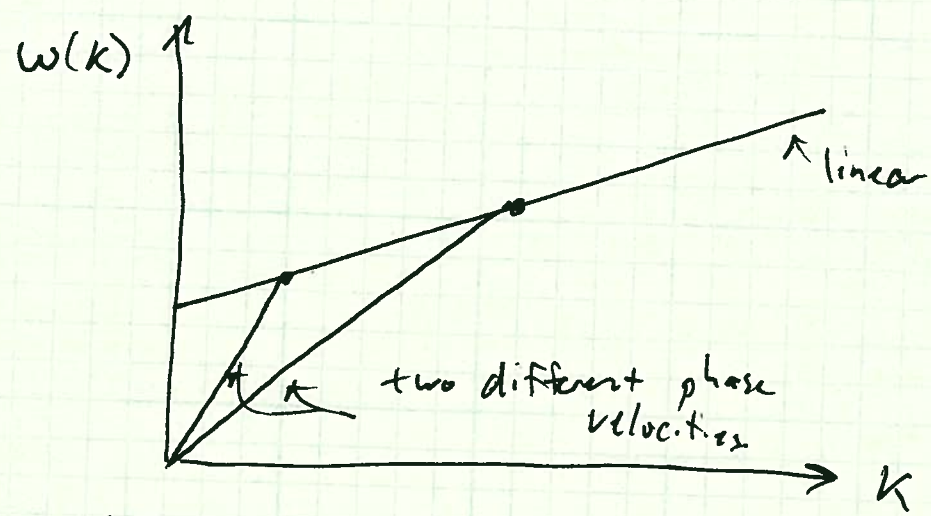
A perfect perfect travelling wave cannot be used to communicate. Because it is a perfect wave, it extends in time to (+) and (−) infinity, and in space to (+) and (−) infinity. To communicate a message, I would need to alter the wave in some way: turn it off, make it larger, change its frequency, etc. But doing any of these things would mean that the wave is no longer perfect, because it would then have multiple frequency components. So to send a message, I will need multiple frequencies at my disposal.

But if the medium is dispersive, then the various frequency components will all travel at different velocities, and my message will disperse. So there will be some limit to how far I can communicate.

However, there is a clever way to send information a much longer distance by using a small range of frequencies to create a pulse.

As long as the dispersion relation is linear over that range of frequencies, we can make a pulse which travels forever.

To illustrate, imagine that our dispersion relation is linear, but not directly proportional:

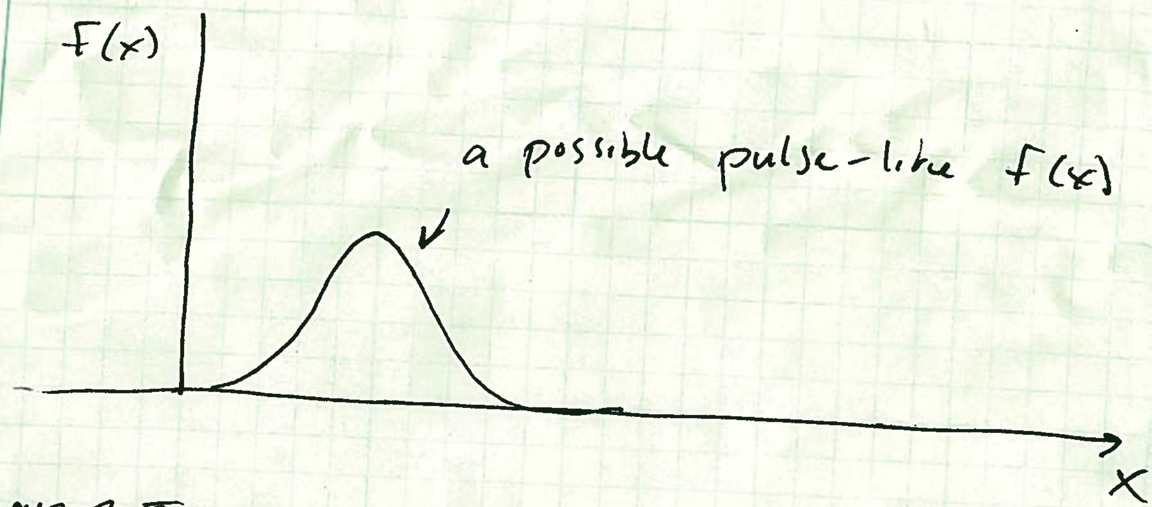


Since different waves have different phase velocities, this system is dispersive.

Now I create a pulse-like "envelope function" composed of a range of wave numbers.

$$f(x) = \text{a pulse-like function} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{F(k)}_{\text{the Fourier transform}} e^{ikx}\, dk$$

f(x) could be a gaussian pulse, for example:



~~Now I multiply~~

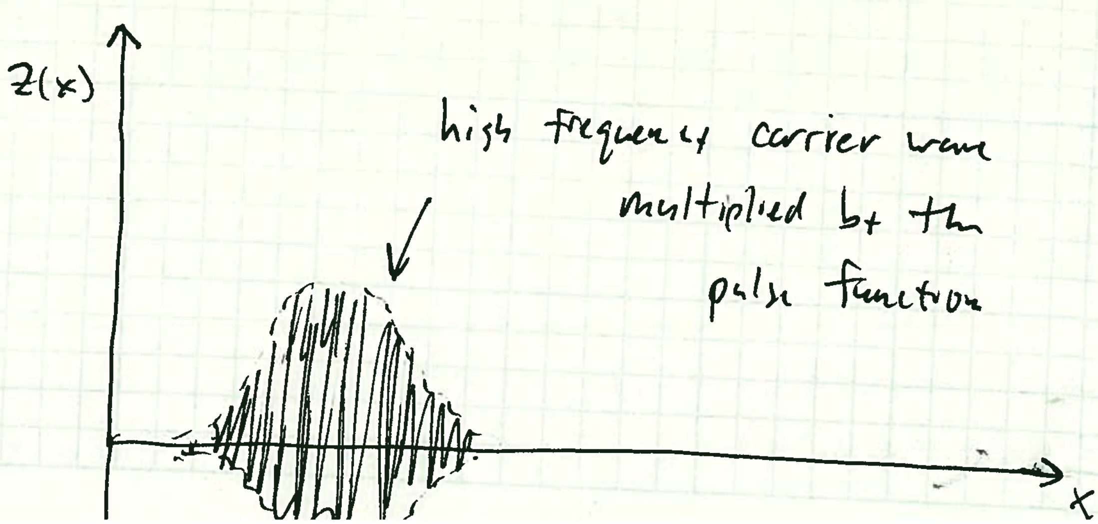
My claim is that I can make this pulse propagate in time forever by multiplying f(x) by a high frequency perfect travelling wave. The high frequency wave is known as the "carrier wave"

so let

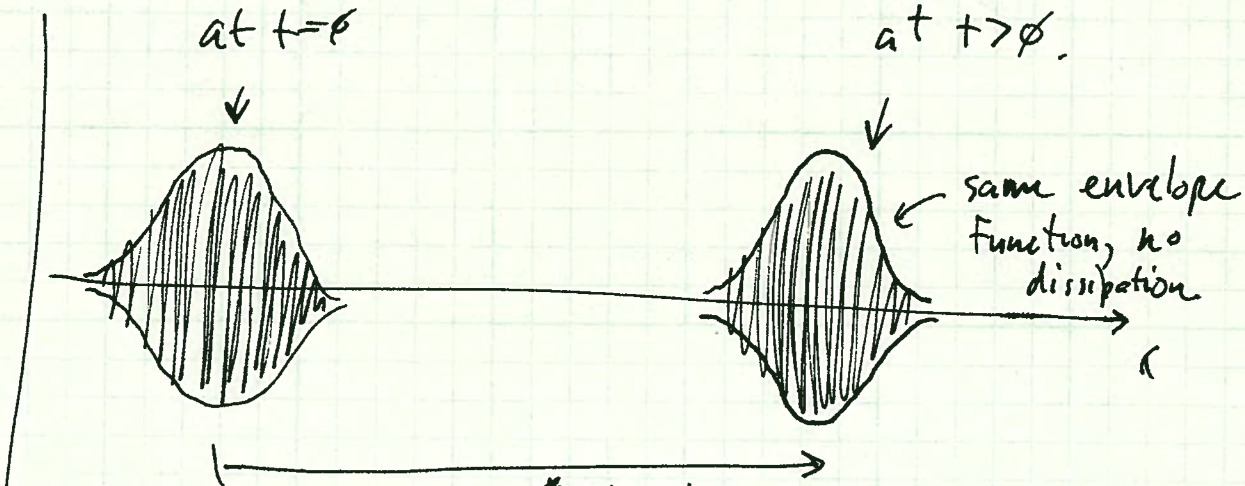
$$Z(x) = (\text{pulse}) \times (\text{carrier}) = f(x)\, e^{ik_c x}$$

where k_c = wave number of the high frequency carrier wave

Now $Z(x)$ looks like



Claim: The Pulse propagates with an envelope function which does not dissipate:



$\Delta x = \underbrace{v_g}\Delta t$

"group velocity"
"the speed at which the envelope propagates".

If this claim is true, then the pulse propagation will be described mathematically as

$$z(x) = f(x)e^{ik_c x} \quad \text{at } t=0$$

we would like to prove this $\rightarrow$ $$\boxed{z(x,t) = f(x-v_g t)\, e^{i(k_c x - \omega_c t)}} \quad \text{at } t>0.$$

v_g: "group velocity"

$\omega_c = k_c v_{phase}$

$f(x - v_g t)$ describes the envelope moving at the group velocity without changing its shape

Now we prove this:

Substitute the Fourier expression for the pulse.

$$z(x) = \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, F(k) e^{ikx}\, dk\right] e^{ik_c x}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, F(k)\, e^{i(k+k_c)x}$$

<u>Trick 1:</u> Let $k' = k + k_c$.

Then $k = k' - k_c$, and we have

$$z(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk'\, F(k'-k_c)\, e^{ik'x}$$

integrate over k' now.

~~This equation says that the Fourier Transform of $z(x)$ is $F(k'-k_c)$.~~

But k' is just a variable of integration. We can re-name it k if we wish:

$$z(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, F(k-k_c)\, e^{ikx} \quad \leftarrow \text{The Fourier Transform expression for } z(x).$$

Now we see that the Fourier Transform of $z(x)$ is $F(k-k_c)$.

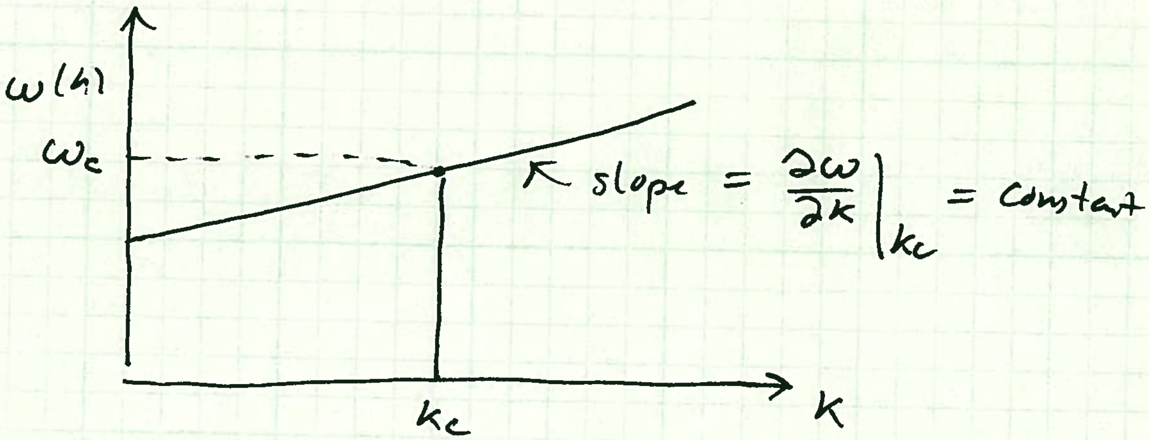
Let's make $z(x)$ move forward in time. To do that we multiply each travelling wave component by $e^{-i\omega(k)t}$:

$$z(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, F(k-k_c)\, e^{ikx}\, e^{-i\omega(k)t}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, F(k-k_c)\, e^{ik\left(x - \frac{\omega(k)}{k}t\right)}$$

Now we use our basic assumption: $\omega(k)$ is linear in k:

$$\omega(k) = \omega_c + (k-k_c)\left.\frac{\partial\omega}{\partial k}\right|_{k_c}$$



We call $\left.\frac{\partial\omega}{\partial k}\right|_{k_c} = v_g =$ "group velocity".

so $\omega(k) = \omega_c + (k-k_c)v_g$.

Then our travelling wave is

$$z(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, F(k-k_c)\, e^{ik\left(x - \left(\frac{\omega_c}{k} + \frac{kv_g}{k} - \frac{k_c v_g}{k}\right)t\right)}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, F(k-k_c)\, e^{ikx}\, e^{-i\omega_c t}\, e^{-ikv_g t}\, e^{ik_c v_g t}$$

~~Trick 2~~

<u>Trick 2:</u> Let $k'' \equiv k-k_c$. Then $k = k''+k_c$. Therefore

$$z(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk''\, F(k'')\, e^{i(k''+k_c)x}\, e^{-i\omega_c t}\, e^{-i(k''+k_c)v_g t} \cdot e^{ik_c v_g t}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk''\, F(k'')\, \underbrace{e^{i(k_c x - \omega_c t)}}_{\text{does not depend on } k''}\, e^{ik''(x - v_g t)}$$

$$z(x,t) = e^{i(k_c x - \omega_c t)} \underbrace{\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk'' F(k'')\, e^{ik''(x - v_g t)}\right]}_{}$$

This is the Fourier Transform of the envelope function $F(x - v_g t)$

$$\boxed{z(x,t) = F(x - v_g t)\, e^{i(k_c x - \omega_c t)}}$$

This is what we set out to prove: the envelope function $F(x)$ propagates without changing its shape: $F(x) \to F(x - v_g t)$. The speed of envelope propagation, known as the group velocity, has been determined to be

$$\boxed{v_g = \text{"group velocity"} = \frac{\partial \omega}{\partial k}(k = k_c)}$$

This result is important because <u>any</u> dispersion relation will be <u>approximately</u> linear over a small range of k. So pulses can be sent through any dispersion medium, as long as we use a sufficiently small range of k to make our pulses.

Example: Quantum free particle