Dispersion

We have been studying systems which are governed by the classical wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v_p^2}\frac{\partial^2 y}{\partial t^2}$$

v_p is the "phase velocity", which is the speed at which a ~~point~~ peak or trough in the wave will travel.

The normal modes are the harmonic solutions to this equation. They are

$$y_n(x) = Ae^{ikx}$$

(recall that when there are no boundary conditions, k is continuous)

with associated frequency

$$\omega = k v_p$$

~~We can~~ Then the time development of the normal mode is

$$y_n(x,t) = \left(Ae^{ikx}\right)e^{i\omega t} = Ae^{i(kx+\omega t)}$$

We can explicitly confirm that this satisfies the equation of motion:

$$\frac{\partial^2 y}{\partial x^2} = -k^2 Ae^{i(kx+\omega t)}$$

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 Ae^{i(kx+\omega t)}$$

Substitute into Eq. of motion:

$$-k^2 A e^{i(kx+\omega t)} \stackrel{?}{=} \frac{1}{v_p^2}\left(-\omega^2 A e^{i(kx+\omega t)}\right)$$

$$-k^2 \stackrel{?}{=} \frac{-\omega^2}{v_p^2}$$

$$v_p = \frac{\omega}{k}.$$

We see that the normal mode $Ae^{i(kx+\omega t)}$ is a solution, as long as the phase velocity is ω/k. We can rewrite this condition.

$$\boxed{\omega = v_p k}$$

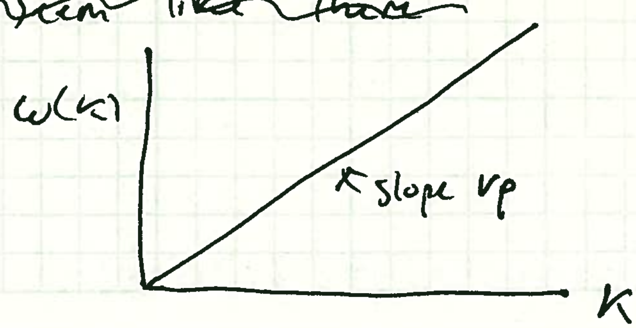
This equation describes all the physics of this system. We can think of this equation as telling us how the normal mode frequency (ω) depends on the normal mode wave number (k):

$$\omega(k) = \underbrace{v_p}\, k$$

$\uparrow$ normal mode frequency depends on the wave number.

$\leftarrow$ linear dependence.

~~It may seem like there~~

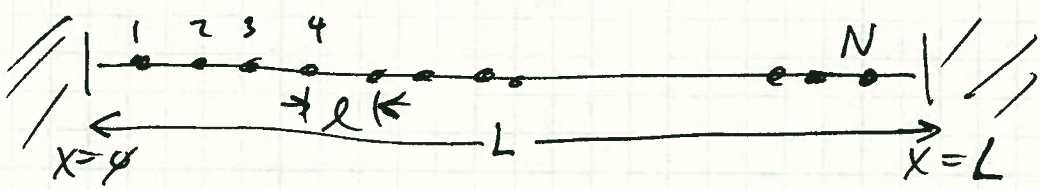


It may seem like there is no other possible relationship between ω and k, but this particular linear relationship only holds true for the classical wave equation. If any other equation of motion is used, then in general there will be some other relationship between ω and k.

An example is the loaded string. For that system, the normal mode frequencies are

$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right)$$

We can re-write this as an $\omega(k)$ function:



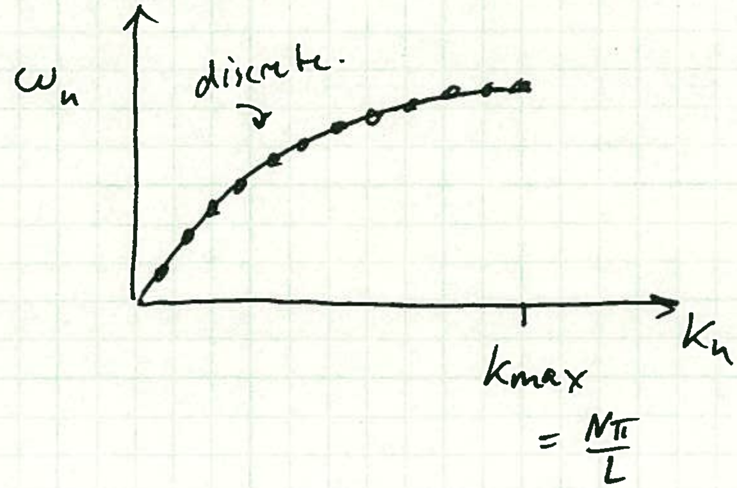
$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi l}{\underbrace{2(N+1)l}_{L}}\right) = 2\omega_0 \sin\left(\frac{n\pi l}{2L}\right)$$

~~now~~ $k_n = \frac{n\pi}{L}$, so we can re-write as

$$\boxed{\omega(k_n) = 2\omega_0 \sin\left(\frac{k_n l}{2}\right)}$$

where n goes from 1 to N.

This is a non-linear relationship between k_n and ω_n



The reason why this system does not have a linear relationship between ω & k is because its equation of motion is not the simple classical wave equation. It's equation is

$$\ddot{y}_p + 2\omega_0^2 y_p - \omega_0^2 (y_{p+1} + y_{p-1}) = \emptyset$$

In general, the relationship between ω & k is determined by the equation of motion, which is determined by the physics of the system.

Notice that when ~~[illegible]~~ $N \to \infty$, the loaded string becomes a continuous ~~[illegible]~~ string ~~[illegible]~~. ~~[illegible]~~ This is equivalent to zooming-in on the linear part of the equation near $k_n = \emptyset$.

For any ~~sta~~ system, the relationship between ω and k is called the "dispersion relation"

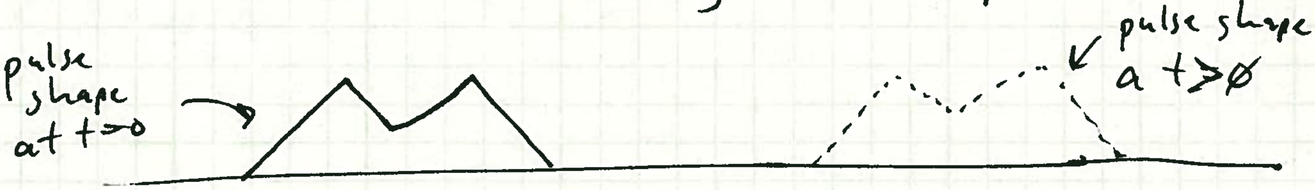
The classical wave equation has a linear dispersion relation:

$$\boxed{\omega(k) = v_p k}$$ ← linear dispersion relation (classical wave equation)

whereas the loaded string has a non-linear dispersion relation:

$$\boxed{\omega(k_n) = 2\omega_0 \sin\left(\frac{k_n l}{2}\right)}$$ ← non-linear dispersion relation (loaded string)

A system which has a linear dispersion relation has a special property: A propagating pulse will travel without changing its shape:



We can show this as follows. The pulse can be described as a sum over normal modes. But the normal modes are continuous, so the sum over normal modes is a Fourier Transform:

$$y(x,t) = \left[\frac{1}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} dk}_{\text{sum over}} \underbrace{A(k)}_{\text{its coefficient}} \underbrace{e^{i(kx)\cancel{\quad}}}_{\text{normal mode}} \underbrace{e^{-i\omega t}}_{\text{its frequency}}\right]$$

$$y(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, A(k) \, e^{i(kx - \omega t)}$$

Now suppose that the system has a linear dispersion relation:

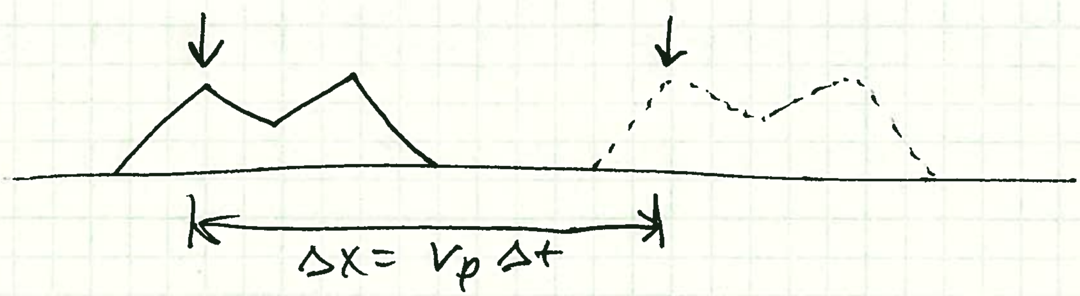
$$\omega = v_p k$$

Then we have

$$y(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, A(k) \, e^{i(kx - (v_p k)t)}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, A(k) \, e^{ik\underbrace{(x - v_p t)}}$$

This says that as time goes forward, if we keep advancing x at speed v_p, then the value of y will stay the same. So the shape of the pulse does not change



Therefore, if $\omega = v_p k$ for the system, then pulses do not disperse. They maintain their shape.

$$\boxed{\text{A linear dispersion relation means that pulses do not disperse.}}$$

This is a special case behavior for systems with linear dispersion relations. But suppose that we have a non-linear dispersion relation. For example, suppose

$\omega \sim k^2$. This happens in quantum mechanics when describing a free particle. In that case,

$$\omega = \frac{\hbar k^2}{2m}.$$

How does a pulse propagate in a system like this?

$$y(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, A(k) e^{ikx} e^{-i\omega t}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk A(k) \, e^{ikx} e^{-i\left(\frac{\hbar k^2}{2m}\right)t}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk A(k) \, e^{ik\underbrace{\left(x - \frac{\hbar k}{2m} t\right)}_{\uparrow}}$$

Here, as time goes forward, we need to advance x at a speed of $v_p = \frac{\hbar k}{2m}$ to keep the argument of the exponential the same.

But the speed $v_p = \frac{\hbar k}{2m}$ is different for every ~~normal~~ wavenumber (k). That is, each normal mode advances at its own velocity, they do not advance together. This means that the various normal modes will disperse; with some travelling fast, and some travelling slowly, and the pulse will disappear.