Normal Mode: In a multi-particle system, a normal mode is a type of motion where all particles oscillate at the same frequency.

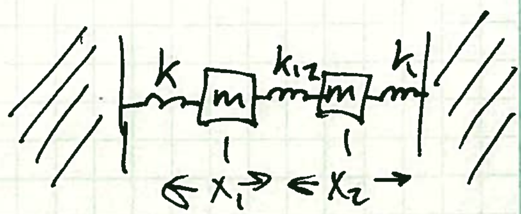
- The number of normal modes is equal to the number of particles.
- Each normal mode goes at its own frequency.
- The general solution is a sum over normal modes:

$$\vec{x}(t) = \sum_{n=1}^{N} a_n \vec{q}_n e^{i\omega_n t}$$

where $\vec{q}_n = (q_{1n}, q_{2n}, \ldots\ldots, q_{Nn}) = n^{th}$ normal mode

= "normal mode eigenvector"

2 coupled oscillators $(N=2)$



Two normal modes:

$$\vec{q}_1 = (1,1) = \text{"symmetric mode"}$$

$$\vec{q}_2 = (1,-1) = \text{"anti-symmetric mode"}$$

Then $\vec{x}(t) = a_1(1,1)e^{i\omega_1 t} + a_2(1,-1)e^{i\omega_2 t}$

$$(x_1(t), x_2(t)) = a_1(1,1)e^{i\omega_1 t} + a_2(1,-1)e^{i\omega_2 t}$$

Eqs. of Motion

$$m\ddot{x}_1 + (k+k_{12})x_1 - k_{12}x_2 = 0$$

$$m\ddot{x}_2 + (k+k_{12})x_2 - k_{12}x_1 = 0$$

For this system we called $\omega_1 = \omega_{small} = \omega_s$

and $\omega_2 = \omega_{Large} = \omega_L$

The frequencies are $\omega_1 = \omega_s = \sqrt{\frac{k}{m}}$

$$\omega_2 = \omega_L = \sqrt{\frac{k + 2k_{12}}{m}}$$

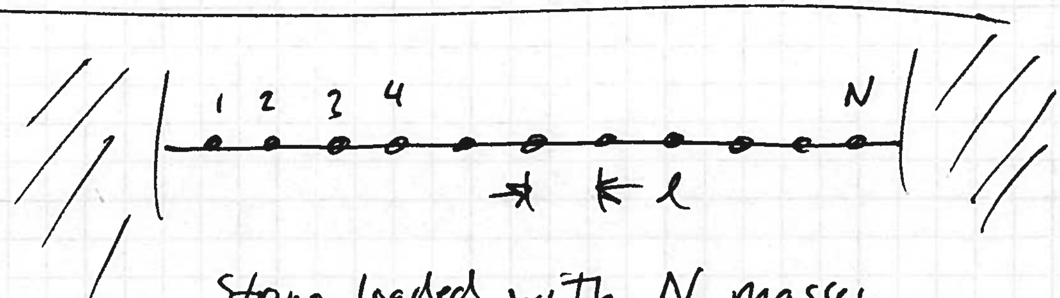
Explicitly the solution is

$$x_1(t) = a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t}$$
$$x_2(t) = a_1 e^{i\omega_1 t} - a_2 e^{i\omega_2 t}$$

or, taking the real part,

$$x_1(t) = a_1 \cos(\omega_1 t) + a_2 \cos(\omega_2 t)$$
$$x_2(t) = a_1 \cos(\omega_1 t) - a_2 \cos(\omega_2 t)$$

N - Coupled oscillators - The loaded string.



String loaded with N masses, each of mass (m).

String tension = T.

Transverse oscillations: Each mass has y-displacement (y_p).

String is fixed at each end, so $y_{p=0} = 0$ and $y_{p=N+1} = 0$. } boundary conditions

Equation of motion for mass (p):

$$\ddot{y}_p + 2\omega_0^2 y_p - \omega_0^2 (y_{p+1} + y_{p-1}) = 0$$

where $\omega_0^2 = T/ml$.

Normal mode solutions:

$$\vec{q}_n = \left(\sin\left(\frac{n\pi}{N+1}\right), \sin\left(\frac{2n\pi}{N+1}\right), \sin\left(\frac{3n\pi}{N+1}\right), \ldots, \sin \frac{Nn\pi}{N+1} \right)$$

$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right)$$

We also wrote the $\vec{q}_n$ vectors on a particle-by-particle basis as

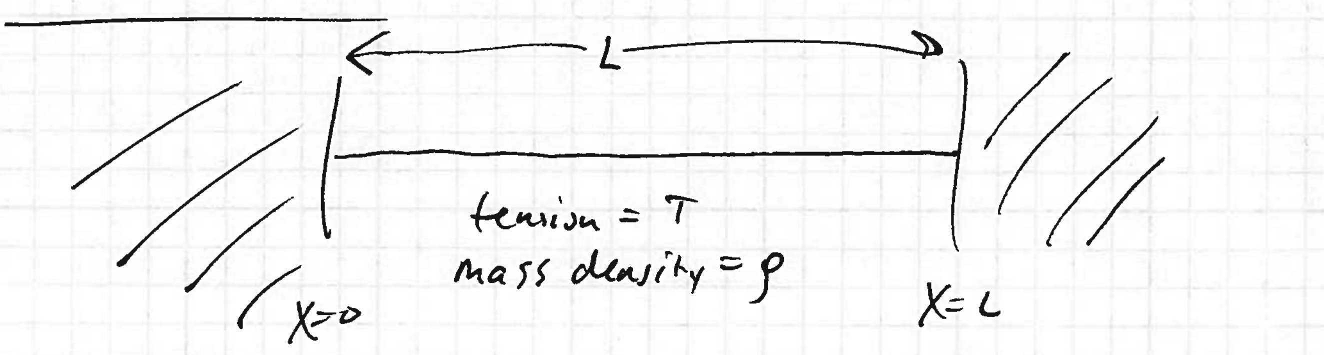
$$A_{pn} = \sin\left(\frac{pn\pi}{N+1}\right)$$

which mass

which normal mode

The solution is the same for longitudinal oscillations, except the motion is in the x direction, rather than the y direction

Continuous Systems - string fixed at $x=0$ & $x=L$



Equation of Motion:

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2}}$$

Classical Wave Equation.

Normal Mode Solution

$$y_n(x) = C_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\omega_n = \sqrt{\frac{T}{\rho}}\,\frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots \infty.$$

General Solution:

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n t}$$

Fourier's Trick

Once the normal modes and normal frequencies of a system are known, the only thing that remains is to find the expansion coefficients to describe the system at $t=0$. We use Fourier's Trick to do this.

For the loaded string, Fourier's Trick says

$$a_i = \frac{\vec{y}_0 \cdot \vec{q}_i}{|\vec{q}_i|^2}$$

where $\vec{y}_0$ is the set of initial positions $(y_1(t=0), y_2(t=0), \ldots\ldots)$

For the continuous string fixed at $x=\emptyset$ and $x=L$, Fourier's Trick says

$$c_n = \frac{2}{L}\int_0^L y(x,t=\emptyset)\,\sin\left(\frac{n\pi x}{L}\right)dx$$

Fourier's Trick depends upon the fact that the eigen vectors are orthogonal:

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{L}{2}\,\delta_{nm}$$ for the continuous string

and $$\sum_{j=1}^{N}\sin\left(\frac{jn\pi}{N+1}\right)\sin\left(\frac{jm\pi}{N+1}\right) = \left(\frac{N+1}{2}\right)\delta_{nm}$$ for the discrete loaded string.

Mathematics of Fourier Series & Fourier Transform

Any periodic function with period $2L$ can be written

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

where
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

This same thing can be written in complex notation:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

where $c_n = \frac{1}{2L}\int_{-L}^{L} F(x)\, e^{-in\pi x/L}\, dx$ ← Fourier's Trick

The two forms can be converted into each other:

$$a_n = c_n + c_{(-n)}$$
$$b_n = i(c_n - c_{(-n)})$$
$$a_0 = c_\emptyset + c_\emptyset = 2c_\emptyset$$

and

$$c_n = \begin{cases} \frac{1}{2}(a_{(-n)} + ib_{(-n)}) & , \text{ for } n < \emptyset \\ \frac{1}{2}a_0 & , \text{ for } n = \emptyset \\ \frac{1}{2}(a_n - ib_n) & , \text{ for } n > \emptyset \end{cases}$$

The complex form is more compact and elegant. It also generalizes to the case where $F(x)$ is <u>no longer periodic</u> (period $L \to \infty$):

$$F(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, A(k)\, e^{ikx} \qquad \text{(non-periodic } F(x)\text{)}$$

$$A(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(x)\, e^{-ikx}\, dx$$ ← "Plancherel's Theorem"

But it's really just another example of Fourier's Trick.

Travelling waves - Continuous String with no boundaries

If we have a continuous string with no boundary conditions (no walls), then we can have travelling wave solutions

$$y(x,t) = A \sin(kx - \omega t)$$

$$k = \frac{2\pi}{\lambda}, \quad \lambda = \text{wavelength}.$$

$$\omega = 2\pi f = \frac{2\pi}{T}$$

The peaks and troughs move forward at the "phase velocity"

$$V_{phase} = v = \frac{\omega}{k} = \lambda f$$

The equation of motion is still the wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T} \frac{\partial^2 y}{\partial t^2}$$

And the travelling waves satisfy this as long as

$$v = \sqrt{\frac{T}{\rho}} = \text{phase velocity}.$$

So we could write the Eq. of Motion as

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

The general solution for a continuous string with no boundaries is a sum over all travelling waves. But since any wavelength is allowed, we have to sum over a continuum of k-values:

$$y(x, t=0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, A(k)\, e^{ikx}$$

where $$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, F(x)\, e^{-ikx}$$

Then, as time goes forward, each normal mode (e^{ikx}) gets its own phase factor $(e^{i\omega t})$

$$y(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, A(k)\, e^{i(kx+\omega t)}$$