Summary of "Fourier's Trick"

Loaded String - N masses

Eigenvectors are given by

$$A_{pn} = \sin\left(\frac{pn\pi}{N+1}\right)$$

which mass

which normal mode

Or, in vector notation,

$$\vec{q}_n = \left(\sin\left(\frac{n\pi}{N+1}\right), \sin\left(\frac{2n\pi}{N+1}\right), \ldots, \sin\left(\frac{Nn\pi}{N+1}\right)\right)$$

These normal mode vectors are <u>orthogonal</u>. We saw this on Homework #7 for the cases of $N=2, 3,$ and 4. In general the orthogonality is mathematically guaranteed by the following trig identity

$$\vec{q}_n \cdot \vec{q}_m = \left(\sin\left(\frac{n\pi}{N+1}\right), \sin\left(\frac{2n\pi}{N+1}\right), \ldots\right) \cdot \left(\sin\left(\frac{m\pi}{N+1}\right), \sin\frac{2m\pi}{N+1}, \ldots\right)$$

$$= \sum_{j=1}^{N} \sin\left(\frac{jn\pi}{N+1}\right) \sin\left(\frac{jm\pi}{N+1}\right) = \left(\frac{N+1}{2}\right)\delta_{nm}$$

trig identity

(I'm not proving this, I'm just invoking a known trig identity.)

Just to be sure, let's put a box around it:

$$\boxed{\sum_{j=1}^{N} \sin\left(\frac{jn\pi}{N+1}\right)\sin\left(\frac{jm\pi}{N+1}\right) = \frac{(N+1)}{2}\delta_{nm}}$$

orthogonality of the loaded string eigenvectors.

How does this compare to the continuous string?

Eigenvectors are continuous functions of x:

$$\text{eigenvector } n = \sin\left(\frac{n\pi x}{L}\right)$$

a continuous vector

They are orthogonal:

$$\boxed{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{L}{2}\delta_{nm}}$$

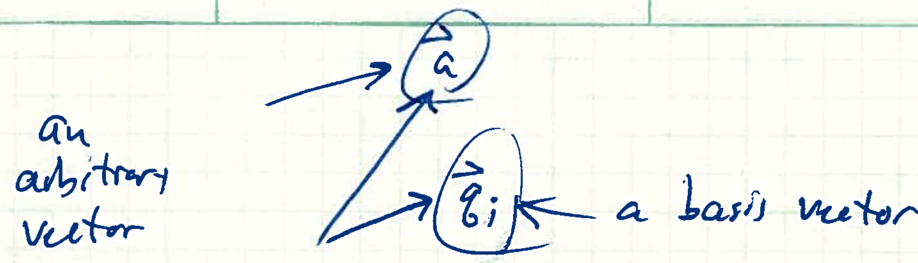
Compare to the discrete case:

$$\boxed{\sum_{j=1}^{N} \sin\left(\frac{jn\pi}{N+1}\right)\sin\left(\frac{jm\pi}{N+1}\right) = \frac{(N+1)}{2}\delta_{nm}}$$

both are dot products

Fourier's Trick:

To ~~as~~ write an arbitrary vector as a sum ~~of~~ of basis vectors, or normal modes, take the dot product with each basis vector:



The component of $\vec{a}$ in the direction of $\vec{q}_i$ is

$$\boxed{a_i = \frac{\vec{a}\cdot\vec{q}_i}{|\vec{q}_i|^2}}$$ Fourier's Trick.

In total, the complete vector $\vec{a}$ is the sum over all ~~the~~ basis vectors:

$$\vec{a} = a_1\vec{q}_1 + a_2\vec{q}_2 + \dots$$

$$= \sum_i a_i\vec{q}_i \qquad \text{where} \quad a_i = \frac{\vec{a}\cdot\vec{q}_i}{|\vec{q}_i|^2}.$$

For the case of continuous vectors, like the normal modes of a continuous string, Fourier Trick says

$$a_n = \underbrace{\left(\frac{2}{L}\right)}_{\text{normalization factor.}} \underbrace{\int_0^L \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{\text{basis vect}} \underbrace{y(x, t=\emptyset)}_{\text{initial condition}}\, dx}_{\text{integral is the dot product}}$$

For a loaded string, with finite eigenvectors, Fourier's Trick says

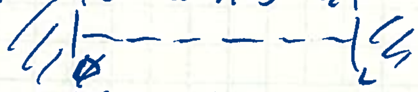
$$a_i = \frac{\overbrace{\vec{y}(t=\emptyset)}^{\text{initial condition}} \cdot \overbrace{\vec{q}_i}^{\text{basis vector}}}{\underbrace{|\vec{q}_i|^2}_{\text{normalization.}}}$$

Generalized Fourier Series

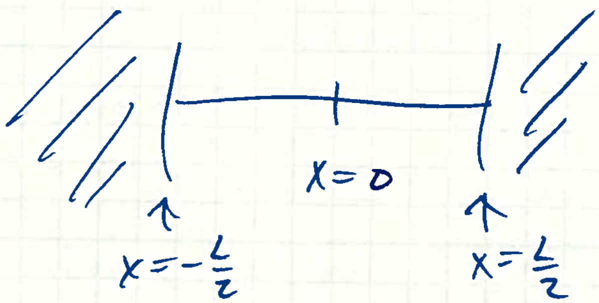
We've been studying a special type of Fourier series called a "<u>Fourier Sine Series</u>"

$$y(x, t=\emptyset) = \sum_{n=1}^{\infty} a \sin\left(\frac{n\pi x}{L}\right)$$

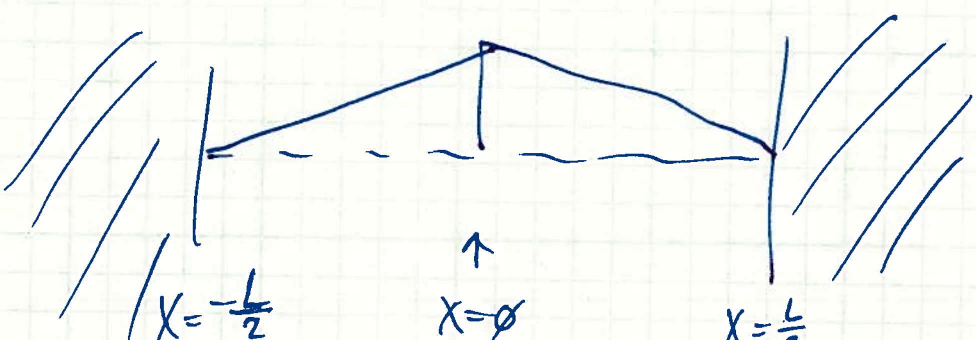
We like this series because it describes a string attached to walls at $x=\emptyset$ & $x=L$.



In general, however, we may choose to attach our string at other locations, like:



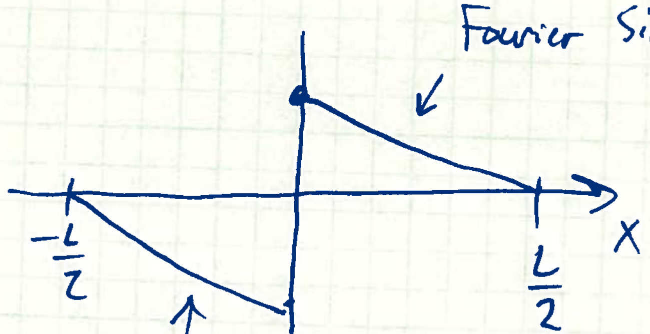
Suppose, for example, that the initial shape of our string is a triangle, and our coordinate system is centered on the middle of the string:



Can we represent this shape as a sum of <u>sine</u> functions?

<u>Answer</u>: No, because sine functions are <u>odd</u> and this function is <u>even</u>. If we tried, we would get

Fourier Sine Series is OK here



but it gives the wrong sign here

It's because every term in the series is <u>odd</u>:

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\therefore f(-x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi(-x)}{L}\right) = -\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

$$= -f(x) \leftarrow \text{odd function}$$

To represent an <u>even</u> function, we'll need a "Fourier Cosine Series"

$$f(x) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \leftarrow \text{"Fourier Cosine Series" for even functions.}$$

How do we determine the expansion coefficients $\{a_n\}$ for this series?

Answer: The cosine functions are orthogonal:

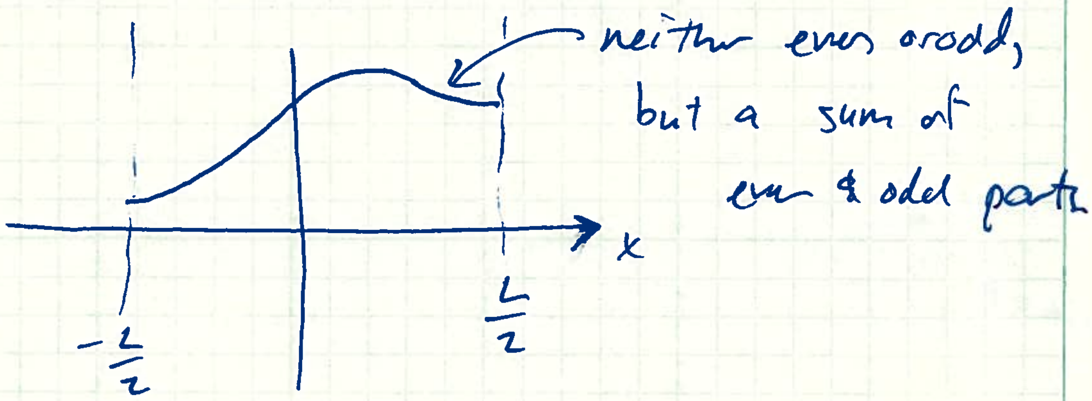
$$\int_{-\frac{L}{2}}^{\frac{L}{2}} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx = \frac{L}{2}\,\delta_{nm}$$

Therefore Fourier's Trick works for them also:

$$a_n = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \cos\left(\frac{n\pi x}{L}\right) f(x)\, dx$$

In general, an arbitrary function is neither even or odd, but is a sum of even and odd parts:

$$f(x) = f_{odd}(x) + f_{even}(x)$$

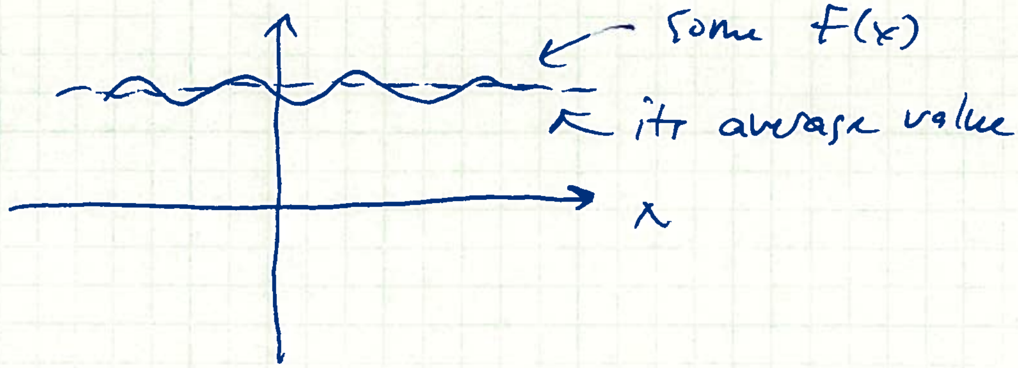


To represent a function like this, we need both Sine & Cosine Terms:

$$f(x) = \sum_n \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

But there's still one thing missing:

If the average value of the function is zero, then sines & cosines are fine. But if the function has a y-offset, then we have to add a constant:

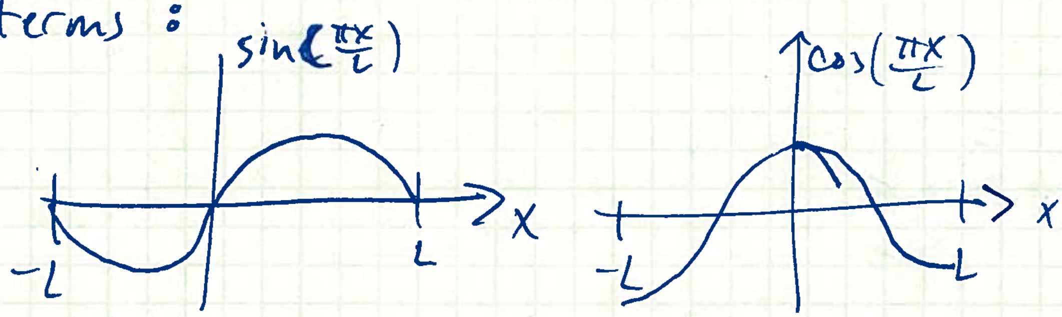


Finally, a complete, general Fourier Series is given by

$$\boxed{f(x) = \frac{a_0}{2} + \sum_n \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]}$$

average value of $f(x)$

This series can represent almost any periodic function. But what is the period? Look at the $n=1$ terms:



The full period is $\boxed{2L}$.

Generalized Fourier Series:

$f(x)$ is ① periodic with period $2L$
② "square integrable" from $-L$ to L.

Then $f(x)$ can be written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Note that normalization factor has changed because now we integrate over a distance of $2L$ instead of L.

Also, note that

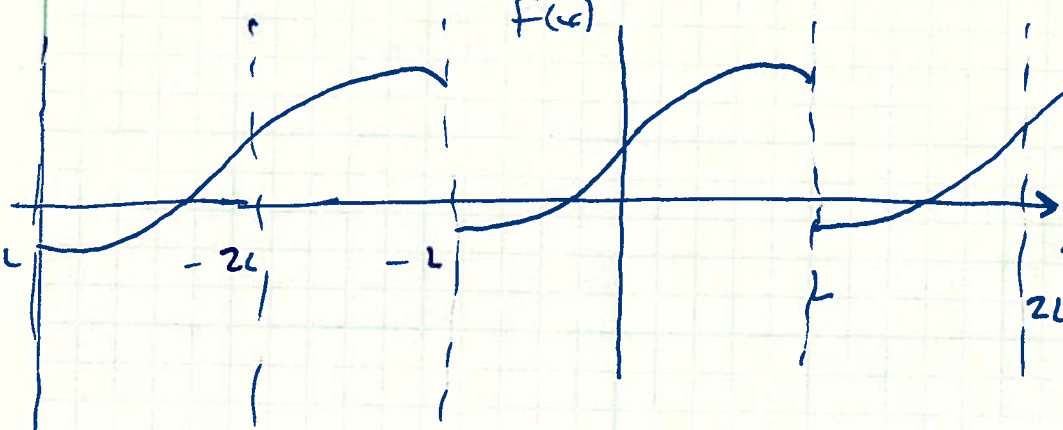
$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{(0)\pi x}{L}\right)dx$$

$$= \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

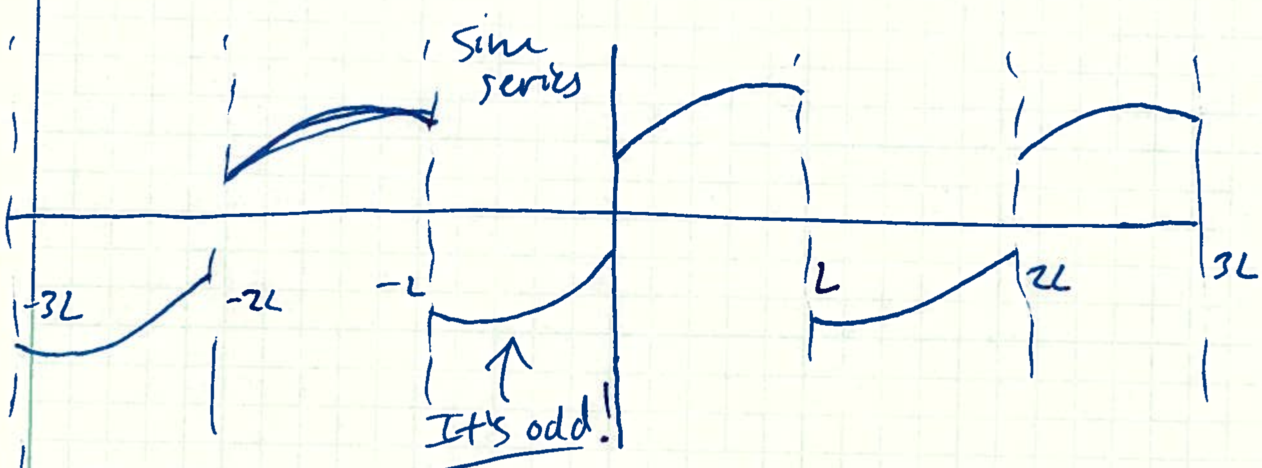
$= 2 \times$ average value of $f(x)$ between $-L$ & L

Picture it:

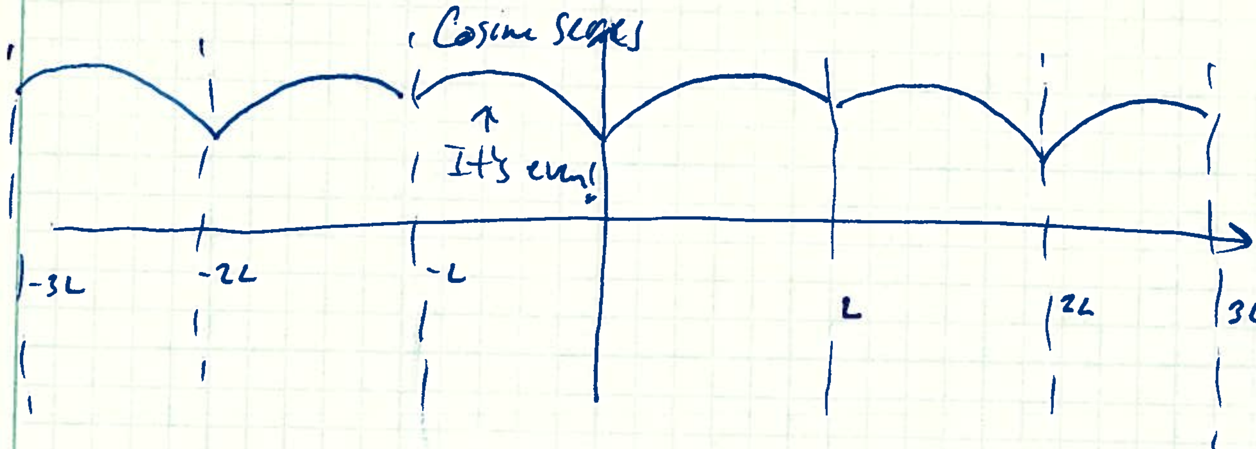
Suppose $F(x)$ looks like this:



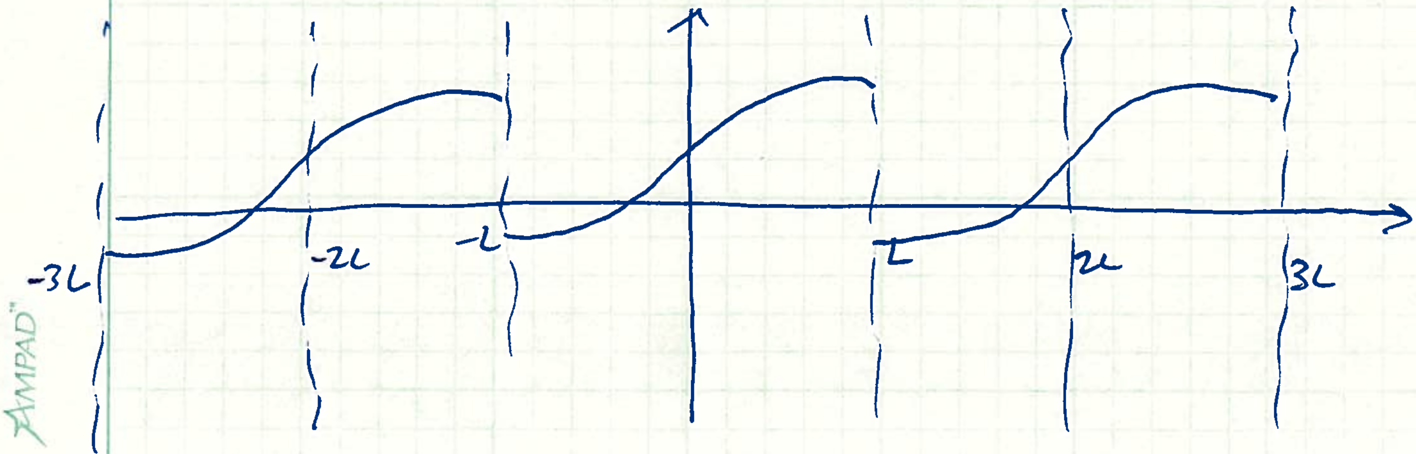
The sine series looks like this:



The cosine series looks like this

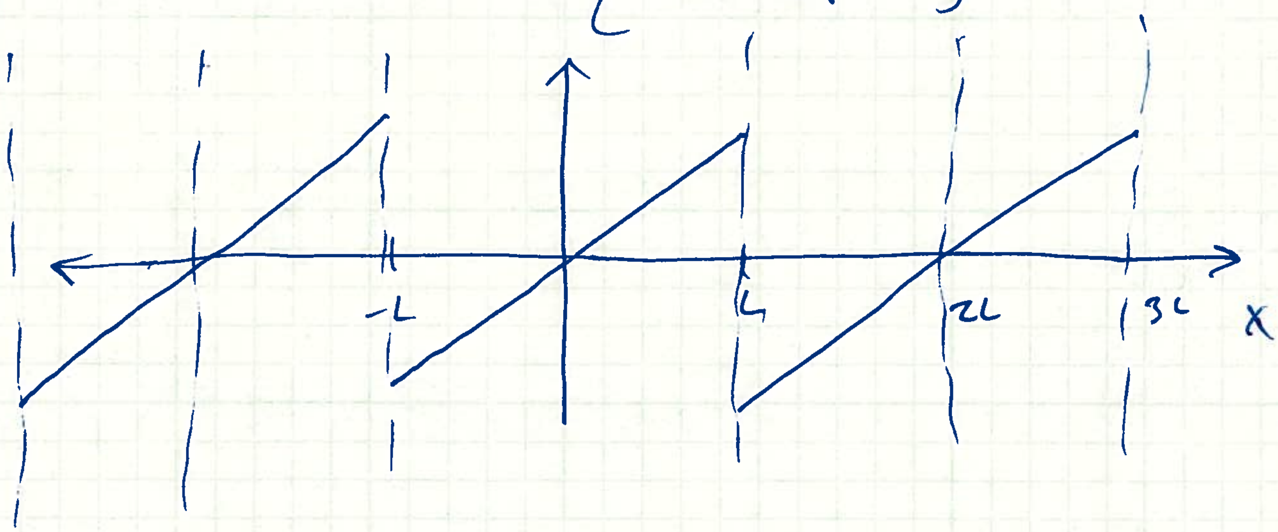


The complete, generalized Fourier Series looks exactly like $f(x)$



Example Calculation of a complete Fourier Series

Sawtooth: $f(x) = \begin{cases} x, \text{ for } -L < x < L \\ \text{and repeating} \end{cases}$



Fourier coefficients:

$$a_0 = \frac{1}{L}\int_{-L}^{L} x\,dx = \emptyset \leftarrow \text{average value is zero}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{x\cos\left(\frac{n\pi x}{L}\right)}_{\text{integrand is odd}}dx = \emptyset \leftarrow \text{no cosine terms! (Function is odd)}$$

$$b_n = \frac{1}{L}\underbrace{\int_{-L}^{L} x \sin\left(\frac{n\pi x}{L}\right) dx}_{\text{look up this integral or integrate by parts}} = -2\left(\frac{L}{n\pi}\right)\cos(n\pi) + 2\left(\frac{L}{n\pi}\right)^2 \cancelto{\phi}{\sin(n\pi)}$$

$$\therefore \quad \boxed{b_n = \frac{2L}{n\pi}(-1)^{n+1}}$$

$$\therefore \quad f(x) = \frac{2L}{\pi}\sin\left(\frac{\pi x}{L}\right) - \frac{2L}{2\pi}\sin\left(\frac{2\pi x}{L}\right) + \frac{2L}{3\pi}\sin\left(\frac{3\pi x}{L}\right) + \cdots$$