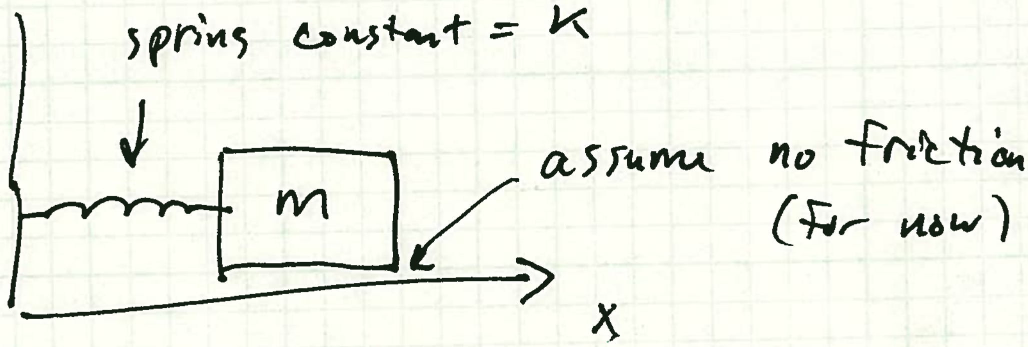
Phys 273 - Lecture 1

Simple Harmonic Oscillator (SHO)

Consider a mass on a horizontal spring:



Force = $\boxed{F = -kx}$ as long as we choose $x = 0$ to be the position where the spring is relaxed and exerts no force.

Hooke's Law (empirical)

minus sign means the force tries to restore the equilibrium.

Equation of Motion: Newton's 2nd Law

$$F = ma \quad , \quad a = \ddot{x} = \frac{d^2x(t)}{dt^2}$$

$$-kx = m\ddot{x}$$

$$\boxed{\ddot{x} + \frac{k}{m}x = 0}$$

Simple Harmonic oscillator Eq. of Motion.

Solution:

$$\boxed{x(t) = A\cos(\omega t + \delta)}$$

Constants:

1. A = "amplitude" = units of meters
   = maximum displacement

2. ω = "angular frequency" = units of Hz
   or seconds^{-1}

3. δ = phase = units of radians

$\Rightarrow$ The argument of the cosine should be in <u>radians</u>, which is unitless.

$\Rightarrow$ The cosine function returns a number between -1 and $+1$ which is <u>unitless</u>.

<u>Question</u>: Why are there 3 constants in our solution?

Answer: Any <u>second order</u> differential equation requires <u>2</u> initial conditions which will specify <u>2</u> of these constants.

For example, ~~specifying~~ specifying the <u>position</u> and <u>velocity</u> at $t = \emptyset$ will fix A & δ.

But what about ω? Can we change ω by using different initial conditions?

<u>Answer</u>: <u>No</u>. ω is determined by k & m.

Substitute the solution into the Eq. of motion:

$$\ddot{x} + \frac{k}{m}x = \phi, \quad \text{where} \quad x(t) = A\cos(\omega t+\delta)$$
$$\ddot{x}(t) = -A\omega^2 \cos(\omega t+\delta)$$

$$-A\omega^2\cos(\omega t+\delta) + \left(\frac{k}{m}\right)A\cos(\omega t+\delta) = \phi$$

$$\therefore \left(\frac{k}{m} - \omega^2\right)A\cos(\omega t+\delta) = \phi$$

How can this equation be true?

① $A = \phi \Rightarrow$ This means that $x(t) = \phi$, the trivial solution.

② $\boxed{\omega = \sqrt{\frac{k}{m}}}$ Simple Harmonic Oscillator Frequency.

All masses on springs vibrate at a frequency $\omega = \sqrt{k/m}$, no matter the initial conditions.

Energy

Potential Energy is stored in the spring:

$$PE(x) = U(x) = -\int_0^x F(x')\,dx' = \int_0^x kx'\,dx' = \frac{1}{2}kx^2$$

integration variable

$$\boxed{U(x) = \frac{1}{2}kx^2} \quad \text{for a spring.}$$

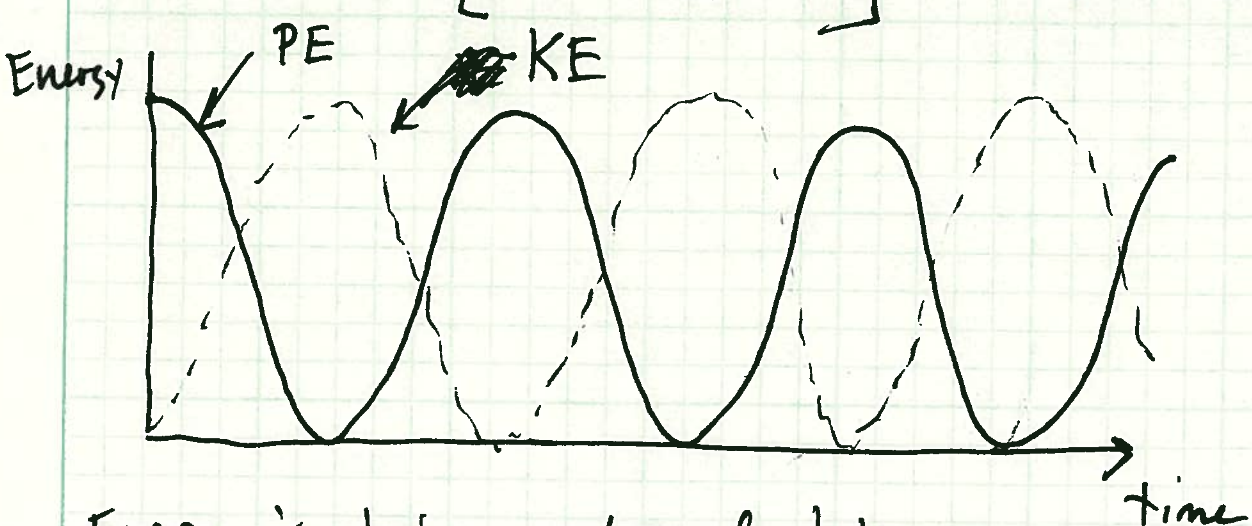
Kinetic Energy

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$$

As a function of time,

$$U(t) = U(x(t)) = \frac{1}{2}k\left[A\cos(\omega t+\delta)\right]^2 = \frac{1}{2}kA^2\cos^2(\omega t+\delta)$$

$$KE(t) = \frac{1}{2}m\left[-A\omega\sin(\omega t+\delta)\right]^2 = \frac{1}{2}mA^2\omega^2\sin^2(\omega t+\delta)$$



Energy is being exchanged between potential and kinetic.

Phys 273 - Lecture 2

Simple Harmonic Oscillator - Mass on a spring

$$\boxed{\ddot{x} + \frac{k}{m}x = \emptyset}$$ Equation of Motion.

Solution: $x(t) = A\cos(\omega t + \delta)$,
where A & δ are determined by initial conditions,
and $\omega = \sqrt{k/m}$.

The oscillator must go at frequency $\omega = \sqrt{k/m}$.

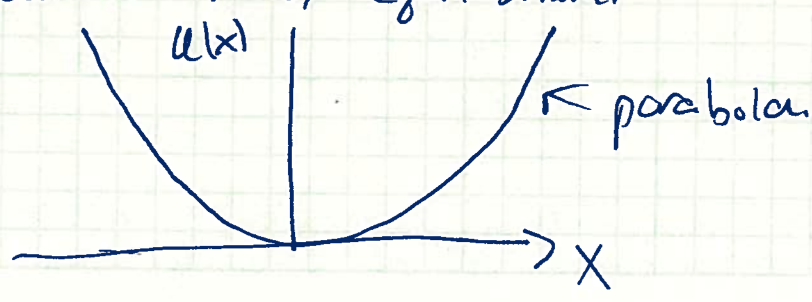
We call this the natural frequency, and use ω_0 to denote it. $\Rightarrow$ $\boxed{\omega_0 \equiv \sqrt{k/m}}$

Then we can write the Eq of Motion as

$$\ddot{x} + \omega_0^2 x = \emptyset.$$

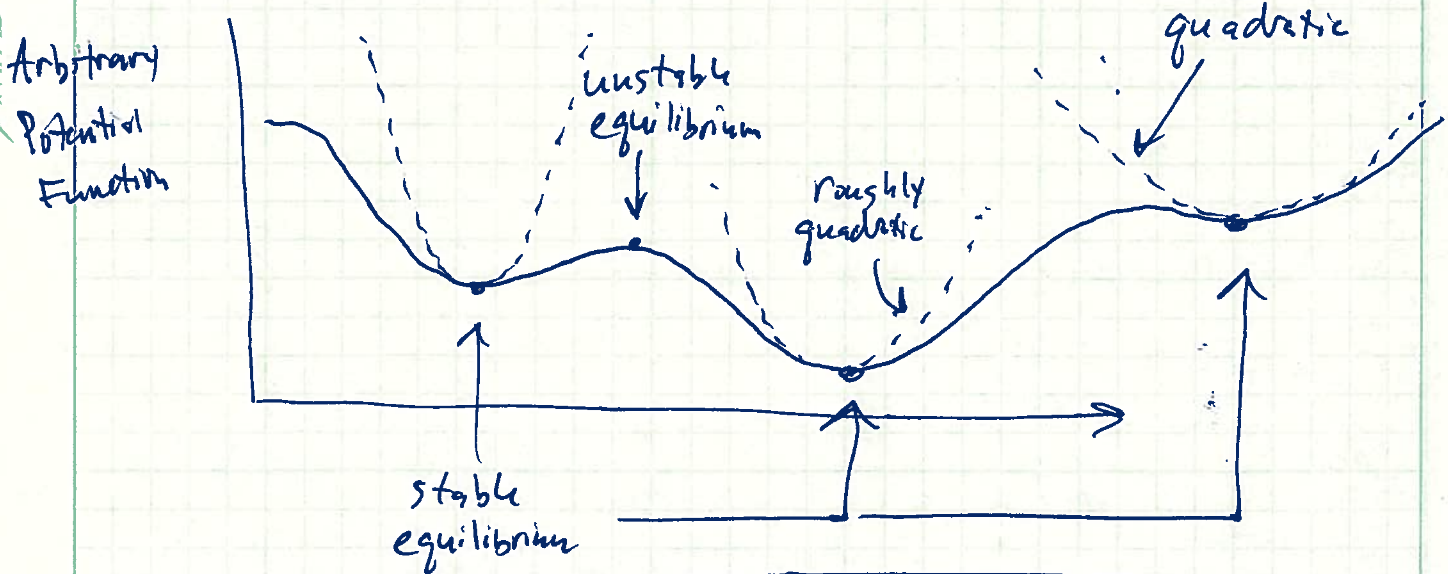
Potential Energy

For a spring $U(x) = \frac{1}{2}kx^2$, where x is the displacement from equilibrium.



$\Rightarrow$ A quadratic potential function gives rise to simple harmonic motion.

This is a very useful result, because almost any potential function is approximately quadratic near a stable equilibrium.



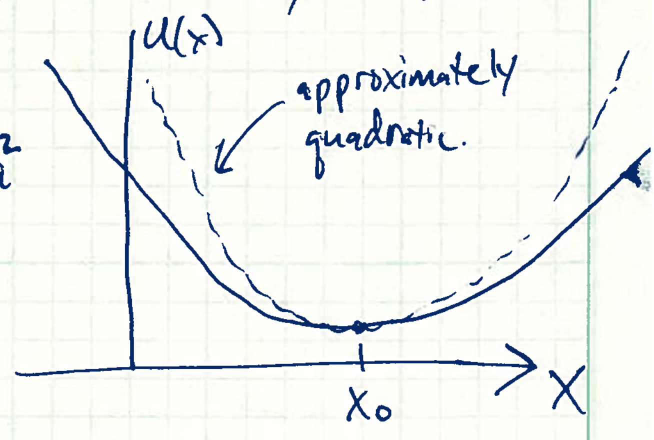
$\Rightarrow$ Small oscillations about stable equilibria tend to be simple harmonic oscillations.

To be formal, expand the potential function about a stable equilibrium in a Taylor Series.

$$U(x_0 + a) = U(x_0) + U'(x_0)\,a + \tfrac{1}{2}U''(x_0)\,a^2 + \ldots$$

equilibrium position (x_0)

displacement from equilibrium (a)



What does this expansion look like for a mass on a spring?

$U(x) = \frac{1}{2}kx^2$ $\quad x_0 =$ equilibrium position $= \emptyset$.

$U(x_0) = \emptyset$

$U'(x_0) = \emptyset \leftarrow$ This must always be true at an equilibrium position, because this is the (negative) force, and the force must be zero at equilibrium.

$U''(x_0) = k$ and $x = x_0 + a = a$ (because $x_0 = \emptyset$).

$\therefore$ $U(x_0+a) = U(a) = \emptyset + (\emptyset)a + \frac{1}{2}ka^2$ + no other term

$x_0 = \emptyset$

$\therefore$ $U(x_0+a) = \frac{1}{2}ka^2 \leftarrow$ exact result for a perfect spring.

Now we know that for a perfect spring we have simple harmonic motion with

$\omega_0 = \sqrt{k/m}$ and $k = U''(x_0)$

or $\boxed{\omega_0 = \sqrt{\frac{U''(x_0)}{m}}}$

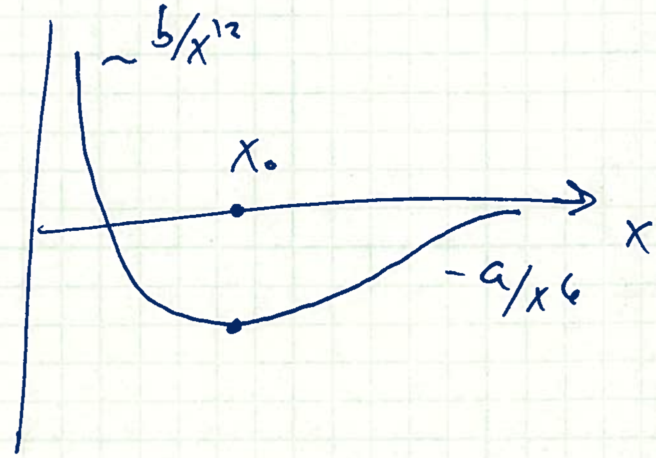
So for a generic potential we will have simple harmonic motion near stable equilibrium with natural frequency

$$\omega_0 \approx \sqrt{\frac{U''(x_0)}{m}}$$

Example Diatomic Molecule

Consider 2 atoms bound in a molecule, one heavy, one light. The potential experienced by the light atom is roughly

$$U(x) = \frac{-a}{x^6} + \frac{b}{x^{12}}$$



$$F(x) = -\frac{dU}{dx} = \frac{1}{x^7}\left(-6a + \frac{12b}{x^6}\right)$$

Equilibrium position: $F(x_0) = 0$

$$6a = \frac{12b}{x_0^6}$$

$$x_0 = \left(\frac{2b}{a}\right)^{\frac{1}{6}}$$

Natural Frequency for small oscillations:

$$U''(x_0) = -\frac{42a}{x_0^8} + \frac{156b}{x_0^{14}} = \frac{-42a\,x_0^6 + 156b}{x_0^{14}} = \frac{-84b + 156b}{x_0^{14}}$$

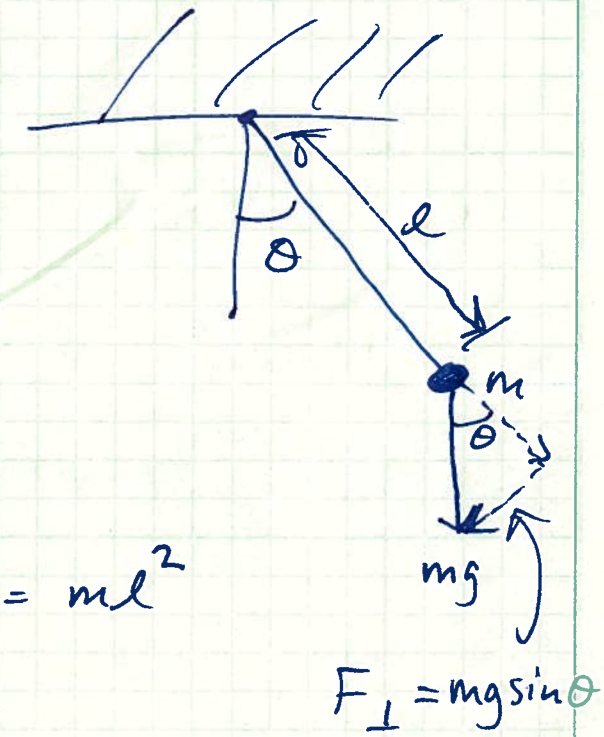
$$= \frac{72b}{x_0^{14}}$$

$$= \frac{72b}{(2b/a)^{7/3}}$$

Then

$$\omega_0 = \sqrt{\frac{U''(x_0)}{m}} = \boxed{\sqrt{\frac{72b}{m(2b/a)^{7/3}}}}$$

Plane Pendulum



Old fashioned way:

torque $= -F_\perp l = -mgl\sin\theta$

Newton's 2^{nd} law:

$\tau = I\ddot{\theta}$

$\uparrow$ moment of inertia $= ml^2$

$$-mgl\sin\theta = ml^2\ddot{\theta}$$

$$\boxed{\ddot{\theta} + \frac{g}{l}\sin\theta = 0}$$ Equation of Motion

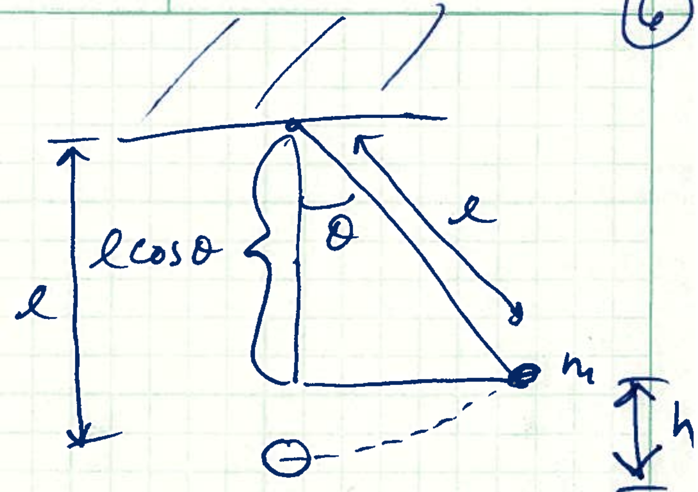
It's not the simple harmonic oscillator equation because of the $\sin\theta$ factor. But for small θ,

$$\sin\theta \approx \theta \quad , \quad \text{so that}$$

$$\boxed{\ddot{\theta} + \frac{g}{l}\theta \approx 0}$$ for small θ.

Then we can read off $\boxed{\omega_0 = \sqrt{g/l}}$ for small oscillations.

More sophisticated way



$$U(x) = mgh$$
$$= mg(\ell - \ell\cos\theta)$$
$$= mg\ell(1-\cos\theta)$$

Then $\dfrac{d^2U}{d\theta^2}\cancel{(\theta=0)} = +mg\ell\cos\theta$

and $\dfrac{d^2U}{d\theta^2}(\theta=\phi) = mg\ell.$

So that $\omega_0 = \sqrt{\dfrac{\frac{d^2U}{d\theta^2}(\theta=\phi)}{\underbrace{I}_{m\ell^2}}} = \sqrt{\dfrac{mg\ell}{m\ell^2}} = \sqrt{\dfrac{g}{\ell}}$