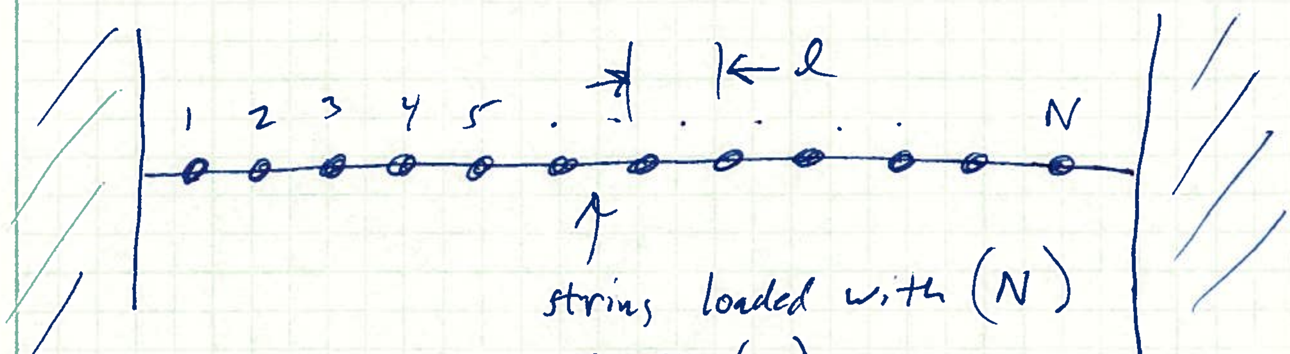
The loaded string: N coupled oscillators and transverse motion.

For a many particle system, the normal modes are easier to visualize if the motion is transverse to the direction of the springs (instead of in the same direction.) So let's switch to transverse motion.

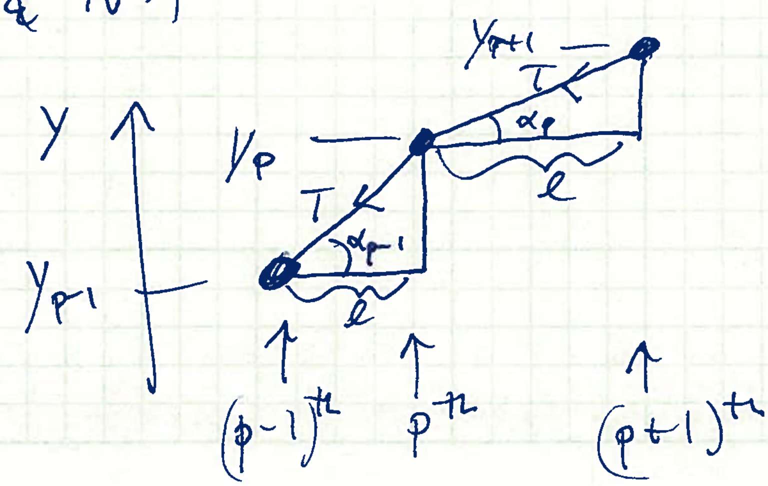


string loaded with (N) masses (m).

Evenly spaced. (distance $= \ell$)

String Tension $= T$

Consider ~~the N^{th}~~ one particular ~~N^{th}~~ mass, let it be the p^{th} mass (something between 1 & N.)



$\ell =$ distance between masses

The neighboring masses exert a restoring force in the y direction through the string tension

$$F_y^{(p)} = -T\sin(\alpha_{p-1}) + T\sin(\alpha_{p+1})$$

~~For small α~~ ($F_x^{(p)}$ will equal zero, otherwise the string will move left or right.)

For small displacements in the y direction, we can approximate the sine function:

$$\sin(\alpha_{p-1}) \approx \frac{y_p - y_{p-1}}{\ell}$$ (because $\sin\theta \approx \tan\theta$ for small θ)

$$\sin(\alpha_p) \approx \frac{y_{p+1} - y_p}{\ell}$$

So $F_y^p \approx -\frac{T}{\ell}(y_p - y_{p-1}) + \frac{T}{\ell}(y_{p+1} - y_p)$

By Newton's 2nd Law, $F_y^p = m\ddot{y}_p$

$$\therefore \quad \ddot{y}_p + \frac{T}{m\ell}(2y_p) - \frac{T}{m\ell}y_{p-1} - \frac{T}{m\ell}y_{p+1} = 0$$

Define $\omega_0^2 \equiv \frac{T}{m\ell}$

$$\boxed{\ddot{y}_p + 2\omega_0^2 y_p - \omega_0^2(y_{p+1} + y_{p-1}) = 0}$$

Equation of motion for the p^{th} mass.

Depends on y_{p+1} & y_p ⇐ Coupled

We will look for normal mode solutions:

$$y_p = A_p e^{i\omega t}$$

normal mode: all masses go at the same frequency.

stopped here 3/6/12 → Our job is to determine:

1. What frequencies ω is this a valid solution? (Normal frequencies.)
2. For each normal frequency, what are the relationships between the amplitudes A_p?

Substitute the guess into the equation of motion:

$$-\omega^2 A_p e^{i\omega t} + 2\omega_0^2 A_p e^{i\omega t} - \omega_0^2 \left(A_{p+1} e^{i\omega t} + A_{p-1} e^{i\omega t}\right) = \emptyset$$

or

$$\left(-\omega^2 + 2\omega_0^2\right) A_p - \omega_0^2 \left(A_{p-1} + A_{p+1}\right) = \emptyset$$

$$\frac{A_{p-1} + A_{p+1}}{A_p} = \underbrace{\frac{-\omega^2 + 2\omega_0^2}{\omega_0^2}}_{\text{constant, independent of } p \text{ (same constant for all masses.)}}$$

Make the following guess:

$$A_p = C \sin(p\theta)$$, where θ is some constant that we must determine.

Does this guess work? Try it:

$$\frac{A_{p-1} + A_{p+1}}{A_p} \overset{?}{=} \text{constant, independent of } p$$

insert guess

$$\frac{C\sin((p-1)\theta) + C\sin((p+1)\theta)}{C\sin(p\theta)} \overset{?}{=} \text{constant}$$

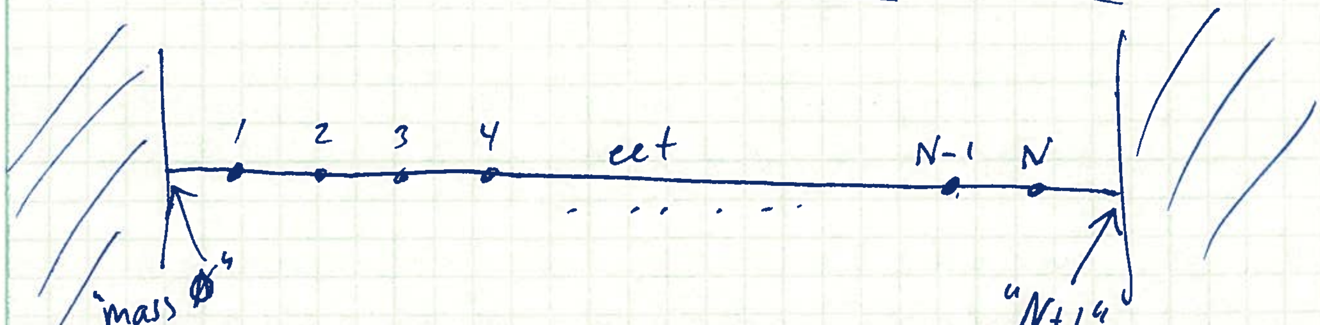
trig identity

$$\frac{2C\sin(p\theta)\cos(\theta)}{C\sin(p\theta)} \overset{?}{=} \text{constant, independent of } p$$

$$\boxed{2\cos(\theta) = \text{constant, independent of } p}$$ ✓ yes

So our guess is viable. But we must determine θ and ω^2.

To fix θ, use the boundary condition:



Because the ends of the string are fixed, the amplitude ~~the~~ A_p should go to zero for $p = \emptyset$ & $p = N+1$. Let's see if that can be made to work

$$A_p = C \sin(p\theta)$$
$$A_\emptyset = C \sin(\emptyset\theta) = \emptyset \checkmark \quad \text{yes}$$

And

$$A_{N+1} = C \sin\left((N+1)\theta\right) = \emptyset$$

$$(N+1)\theta = n\pi, \quad n = 1, 2, 3, 4, \ldots$$

$$\boxed{\theta = \frac{n\pi}{N+1}, \; n = 1, 2, 3}$$

Therefore our solution for A_p is

$$\boxed{A_p = C \sin\left(\frac{pn\pi}{N+1}\right)}$$

What about the normal frequencies?

$$\underbrace{\frac{A_{p-1} + A_{p+1}}{A_p}}_{2\cos(\theta)} = \frac{-\omega^2 + 2\omega_0^2}{\omega_0^2}$$

$$2\cos(\theta) = \frac{-\omega^2 + 2\omega_0^2}{\omega_0^2}$$

$$\theta = \frac{n\pi}{N+1}$$

$$2\cos\left(\frac{n\pi}{N+1}\right) = \frac{-\omega^2 + 2\omega_0^2}{\omega_0^2}$$

$$\omega^2 = 2\omega_0^2\left(1 - \cos\left(\frac{n\pi}{N+1}\right)\right)$$

trig identity

$$\omega^2 = 4\omega_0^2 \sin^2\left(\frac{n\pi}{2(N+1)}\right)$$

$$\boxed{\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right)}$$

Normal Frequencies.

Add an (n) subscript, because RHS depends on (n)

So we have found the normal mode solutions. The amplitude relationship is

$$\boxed{\begin{aligned} A_{pn} &= C \sin\left(\frac{pn\pi}{N+1}\right) \\ \omega_n &= 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right) \end{aligned}}$$

Normal Modes for a string loaded with N masses.

and the normal frequencies are:

In this expression:

- p is an integer which tells us which mass we are talking about
- N is the number of masses ($p = 1, 2, \dots, N$)
- n tells us which ~~of~~ ~~[illegible]~~ normal mode we are considering.
- $\omega_0 = T/ml$

Properties of the Normal Modes of the loaded string

Recall that the displacement of the p^{th} mass for a particular normal mode is

$$y_p = A_{pn} e^{i\omega t} = C \sin\left(\frac{pn\pi}{N+1}\right) e^{i\omega t}$$

Here we have assumed that the phase at $t = 0$ is zero. If we want to allow a non-zero phase we could write

$$y_p = A_{pn} e^{i(\omega t + \delta)} \quad \text{or}$$

$$\text{or} \quad y_p = \underbrace{\left(A_{pn} e^{i\delta}\right)}_{B_{pn}} e^{i\omega t}.$$

where B_{pn} is complex.

Also, the allowed frequencies are

$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right)$$

Question: How many normal modes are there?

Answer: For a system of N masses, there are N normal modes.

We can see this as follows:

$$\omega_{N+2} = 2\omega_0 \sin\left(\frac{(N+2)\pi}{2(N+1)}\right) = 2\omega_0 \sin\left[\frac{(2(N+1) - N)\pi}{2(N+1)}\right]$$

$$= 2\omega_0 \sin\left[\pi - \frac{N\pi}{2(N+1)}\right]$$

trig identity

$$= 2\omega_0 \sin\left(\frac{N\pi}{2(N+1)}\right)$$

$$= \omega_N$$

ω_{N+2} just duplicates ω_N ... it is not an independent solution.

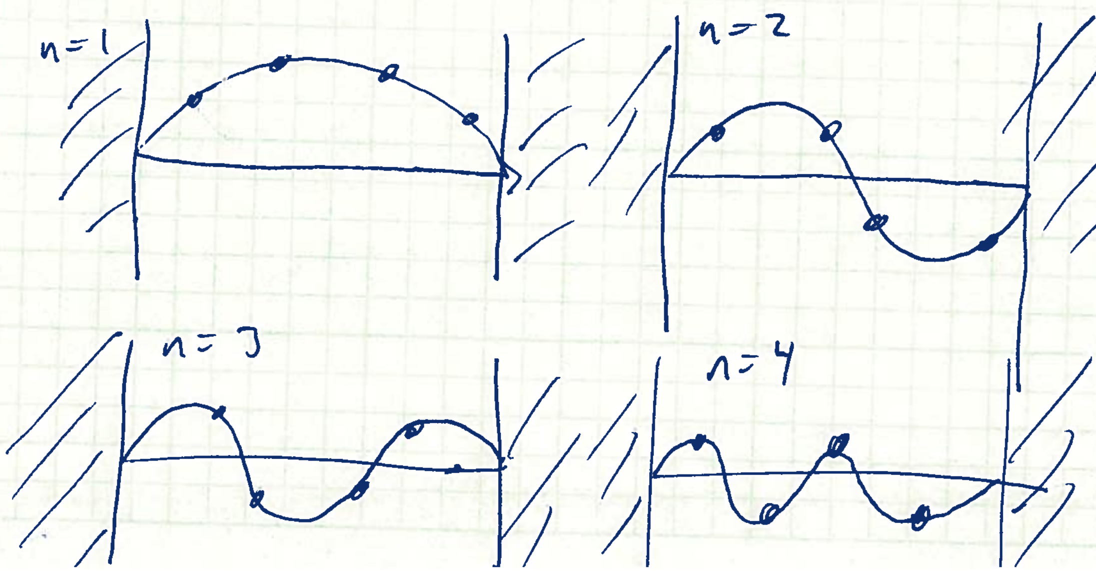
Similarly, $\omega_{N+3} = \omega_{N-1}$

It can also be shown that the amplitude relationship (A_p) repeats itself when $n > N$.

Conclusion: There are N independent normal modes of a system of N masses on a string.

What do the modes look like?

For $N = 4$:



The general solution is a superposition of normal modes:

$$y_p = \sum_{n=1}^{N} a_n \sin\left(\frac{pn\pi}{N+1}\right) e^{i\omega_n t}$$

amplitude factor.

time evolution

complex coefficient, 2 free parameters each

The number of free parameters is $2N$: real & imaginary parts of a_n, where $n = 1, 2, \ldots N$. These $2N$ free parameters will be fixed by the $2N$ initial conditions: the position & velocity of every particle at $t = \phi$.

We can also switch to vector notation (if we like). Let

$$\vec{y} = (y_1, y_2, y_3, \ldots, y_N)$$

$$\vec{q}_n = \left(\sin\left(\frac{n\pi}{N+1}\right), \sin\left(\frac{2n\pi}{N+1}\right), \sin\left(\frac{3n\pi}{N+1}\right), \ldots, \sin\frac{Nn\pi}{N+1}\right)$$

Then $\vec{y} = \sum_{n=1}^{N} a_n \vec{q}_n e^{i\omega_n t}$

coefficient of the nth mode

time evolution

normal mode vector

(amplitude relationship)

Initial Conditions and Fourier's Trick

"Fourier's Trick" (terminology from David Griffith's Quantum Mechanics book), is a way to determine the expansion coefficients $\{a_n\}$ while doing very little work. It allows you to get the answer right away, given the initial conditions. It relies on the following observation:

> The eigenvectors which describe the normal modes of the loaded string are orthogonal to each other.

Recall: $$q_n = \left(\underset{\text{mass \#1}}{\sin\left(\frac{n\pi}{N+1}\right)},\ \underset{\text{mass \#2}}{\sin\left(\frac{2n\pi}{N+1}\right)},\ \sin\left(\frac{3n\pi}{N+1}\right), \ldots,\ \underset{\text{mass \#N}}{\sin\left(\frac{Nn\pi}{N+1}\right)} \right)$$

Illustration: Consider $N=2$.

$$q_1 = \left(\underset{p=1}{\sin\frac{\pi}{3}},\ \underset{p=2}{\sin\frac{2\pi}{3}}\right) = (0.866, 0.866) \leftarrow \text{symmetric mode}$$

$$q_2 = \left(\sin\frac{2\pi}{3},\ \sin\frac{4\pi}{3}\right) = (0.866, -0.866)$$

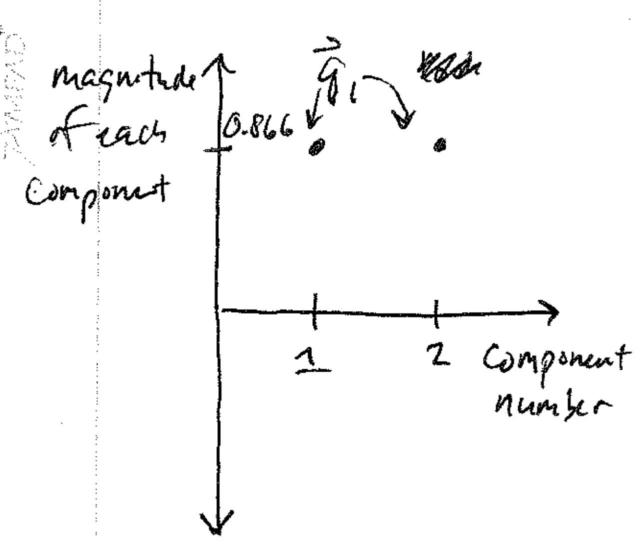
To show that they are orthogonal, take the dot product:

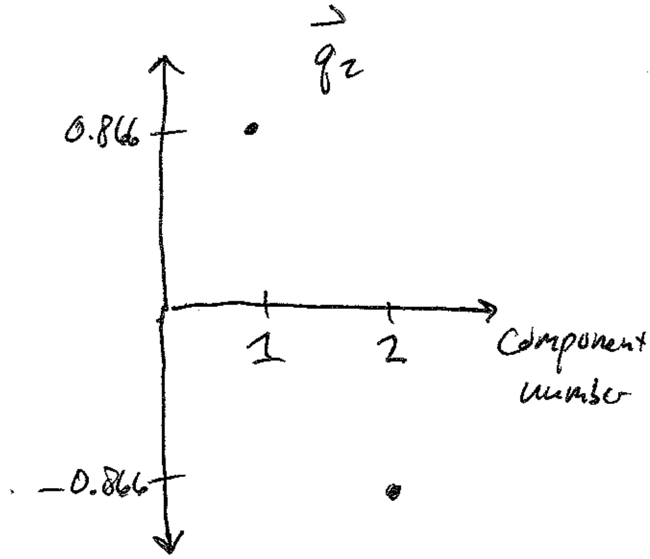
$$\vec{q}_1 \cdot \vec{q}_2 = \Big((0.866)(0.866) + (0.866)(-0.866) \Big) \boxed{= \phi}$$

Orthogonal.

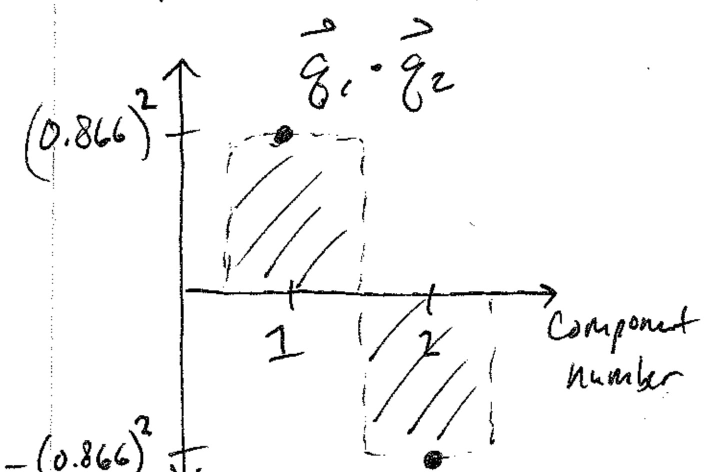
When the dot product is zero, the vectors are orthogonal.

Let's draw this:





To visualize the dot product, multiply the two graphs component-by-component and add them all up:



Visualize the sum by adding the areas under the multiplied components.

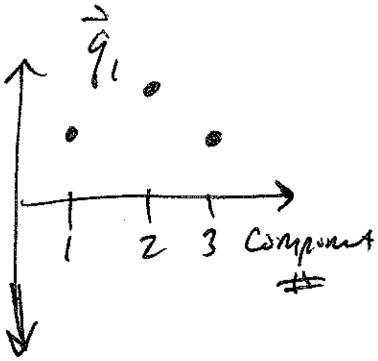
These add to zero, and the area is zero (because the 2nd component is negative).

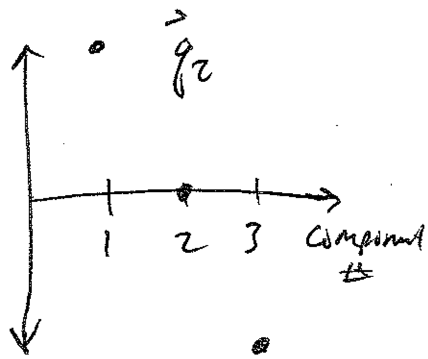
Illustration: Consider $N=3$

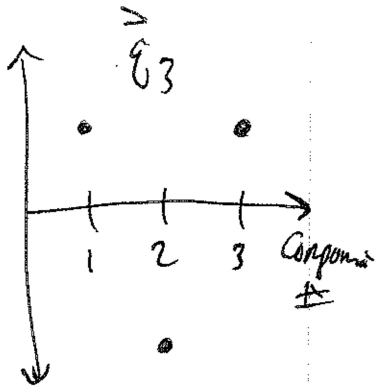
$$\vec{q}_1 = \left(\sin\frac{\pi}{4}, \sin\frac{2\pi}{4}, \sin\frac{3\pi}{4}\right) = (0.707, 1, 0.707)$$

$$\vec{q}_2 = \left(\sin\frac{2\pi}{4}, \sin\frac{4\pi}{4}, \sin\frac{6\pi}{4}\right) = (1, 0, -1)$$

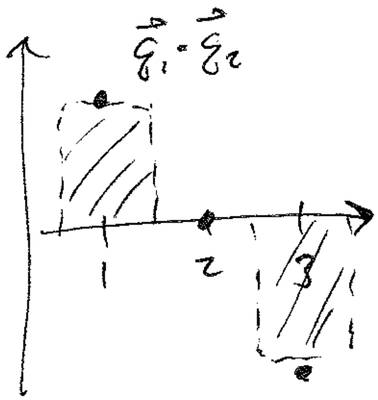
$$\vec{q}_3 = \left(\sin\frac{3\pi}{4}, \sin\frac{6\pi}{4}, \sin\frac{9\pi}{4}\right) = (0.707, -1, 0.707)$$







Try $\vec{q}_1 \cdot \vec{q}_2$:

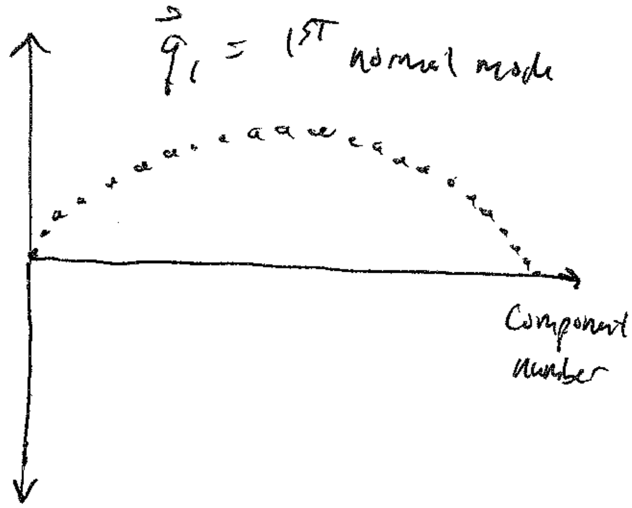


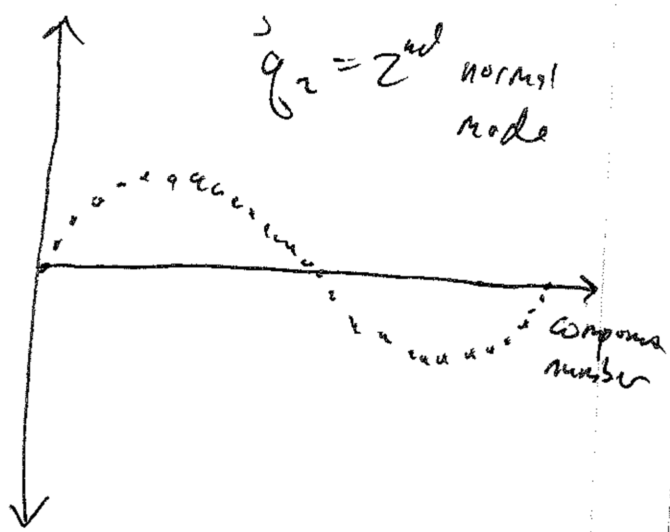
← Areas sum to zero, so $\vec{q}_1 \cdot \vec{q}_2 = \emptyset$.

The same thing happens for $\vec{q}_2 \cdot \vec{q}_3 = \emptyset$,
$\vec{q}_1 \cdot \vec{q}_3 = \emptyset$.

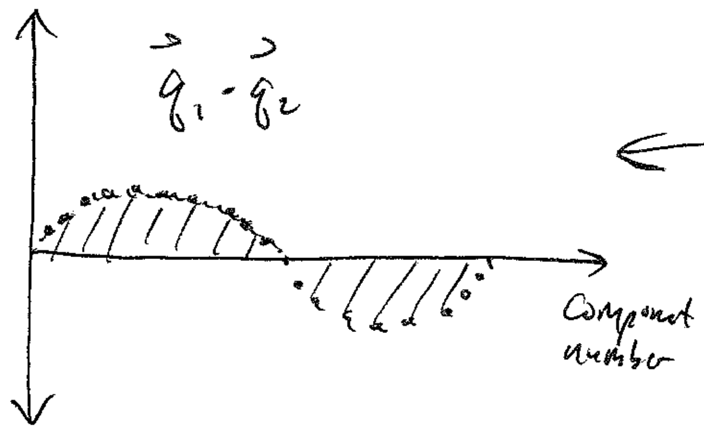
Illustration: Consider the loaded string with a large number of masses: $N =$ large.

Let's draw the 1st and 2nd normal modes:





What does the dot product look like?

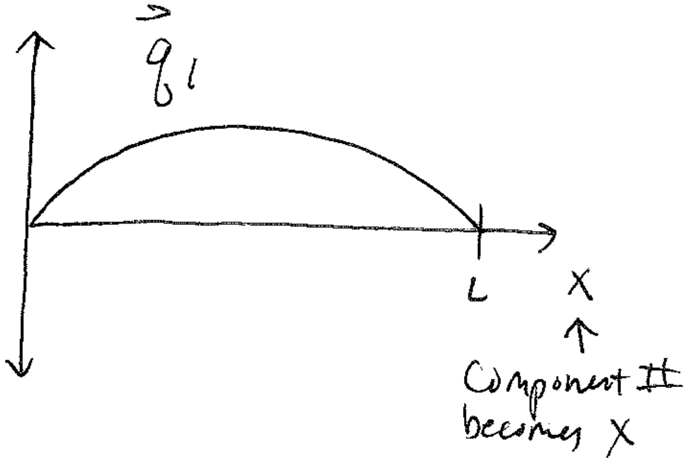


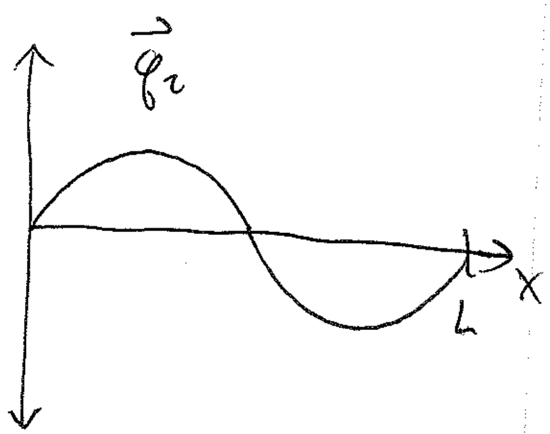
← Area $= \phi$, so $\vec{q}_1 \cdot \vec{q}_2 = \phi$.

Illustration: The continuous case: $N \to \infty$.

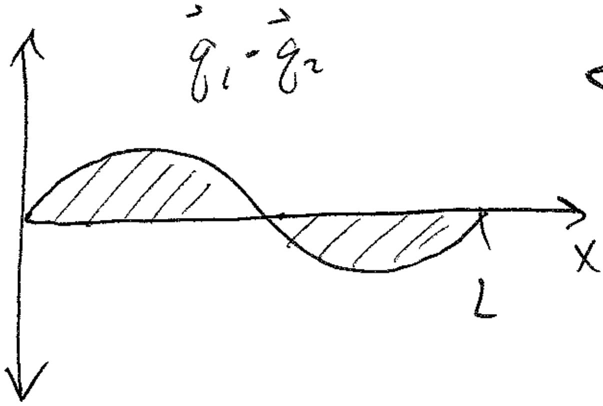
$\vec{q}_1$ = normal mode 1 = $\sin\left(\frac{\pi x}{L}\right)$

$\vec{q}_2$ = normal mode 2 = $\sin\left(\frac{2\pi x}{L}\right)$





The dot product looks like



← Area = ∅, so they are orthogonal.

Mathematical Statements:

Eigenvectors of the loaded string are orthogonal:

$$\vec{q}_n \cdot \vec{q}_m = \left(\sin\left(\frac{n\pi}{N+1}\right), \sin\left(\frac{2n\pi}{N+1}\right), \ldots\right) \cdot \left(\sin\left(\frac{m\pi}{N+1}\right), \sin\left(\frac{2m\pi}{N+1}\right) \ldots\right)$$

$$= \sum_{p=1}^{N} \sin\left(\frac{pn\pi}{N+1}\right) \sin\left(\frac{pm\pi}{N+1}\right) = \begin{cases} \left(\frac{N+1}{2}\right), & \text{if } n=m \\ \emptyset, & \text{if } n \neq m \end{cases}$$

In this last step I'm invoking a known trig identity (I am not proving it here.)

To simplify the notation, let

$$\delta_{nm} \equiv \text{"Kronecker Delta"} \equiv \begin{cases} 1, & n=m \\ \emptyset, & n \neq m \end{cases}$$

Then we can say that

$$\boxed{\vec{q}_n \cdot \vec{q}_m = \sum_{p=1}^{N} \sin\left(\frac{pn\pi}{N+1}\right) \sin\left(\frac{pm\pi}{N+1}\right) = \left(\frac{N+1}{2}\right) \delta_{nm}}$$

or even

$$\boxed{\vec{q}_n \cdot \vec{q}_m = \left(\frac{N+1}{2}\right) \delta_{nm}}$$

This says that $\vec{q}_n$ and $\vec{q}_m$ are orthogonal: their dot product is zero if they are different eigenvectors.

For the continuous case, the mathematical statement is:

eigenvector $n = \sin\left(\frac{n\pi x}{L}\right)$

↑ a continuous vector, a function of a continuous variable x.

Statement of orthogonality:

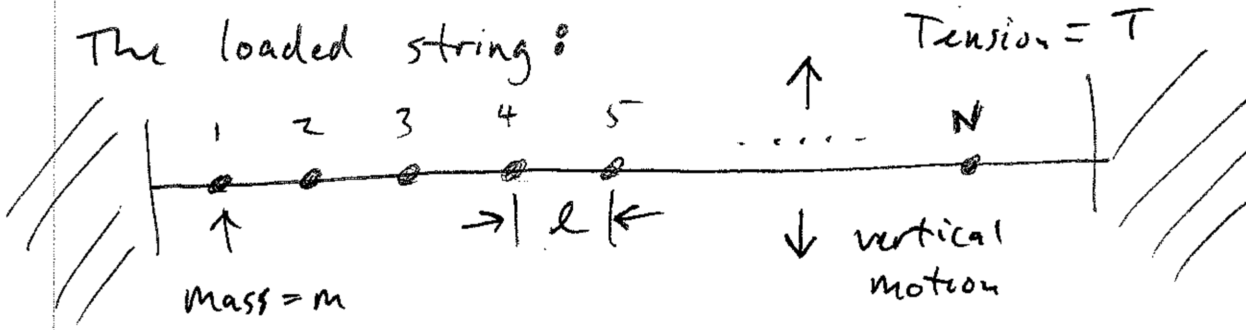
$$\boxed{\underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx}_{\text{a continuous dot product}} = \frac{L}{2} \delta_{nm}}$$

you will prove this on the next homework.

This says that the continuous vectors $\sin\left(\frac{n\pi x}{L}\right)$ and $\sin\left(\frac{m\pi x}{L}\right)$ are orthogonal: their dot product is zero if $n \neq m$.

Continuous Systems - Wave Equation.

The loaded string:



Equation of Motion: y_p = y position of mass # p.

$$\boxed{\ddot{y}_p + 2\omega_0^2 y_p - \omega_0^2 (y_{p+1} + y_{p-1}) = \phi}$$

where $\omega_0^2 = \dfrac{T}{ml}$

Normal Mode Solutions:

$$\boxed{\begin{array}{l} y_p = C \sin\left(\dfrac{pn\pi}{N+1}\right) e^{i\omega_n t} \\ \\ \text{where} \quad \omega_n = 2\omega_0 \sin\left(\dfrac{n\pi}{2(N+1)}\right) \end{array}}$$

and n = the mode number = 1, 2, ..., N

General Solution is a sum over normal modes:

$$y_p = \underbrace{\sum_{n=1}^{N}}_{\text{sum over normal modes}} C_n \sin\left(\frac{pn\pi}{N+1}\right) \underbrace{e^{i\omega_n t}}_{\text{time evolution}}$$

expansion coefficients, determined by the initial conditions.

Write the general solution in vector notation:

Let $\vec{y} = (y_1, y_2, y_3, \ldots, y_N)$

$$\vec{q}_n = \left(\sin\left(\frac{n\pi}{N+1}\right), \sin\left(\frac{2n\pi}{N+1}\right), \sin\left(\frac{3n\pi}{N+1}\right), \ldots, \sin\left(\frac{Nn\pi}{N+1}\right) \right)$$

Then the general solution is

$$\boxed{\vec{y} = \sum_{n=1}^{N} c_n \vec{q}_n e^{i\omega_n t}}$$

Because the eigenvectors $\{q_n\}$ are orthogonal, the expansion coefficients can be calculated by taking the dot product with the initial conditions vectors:

Let $\vec{y}_0$ = initial position vector

$\vec{v}_0$ = initial velocity vector

And let $c_n \equiv a_n + i b_n$

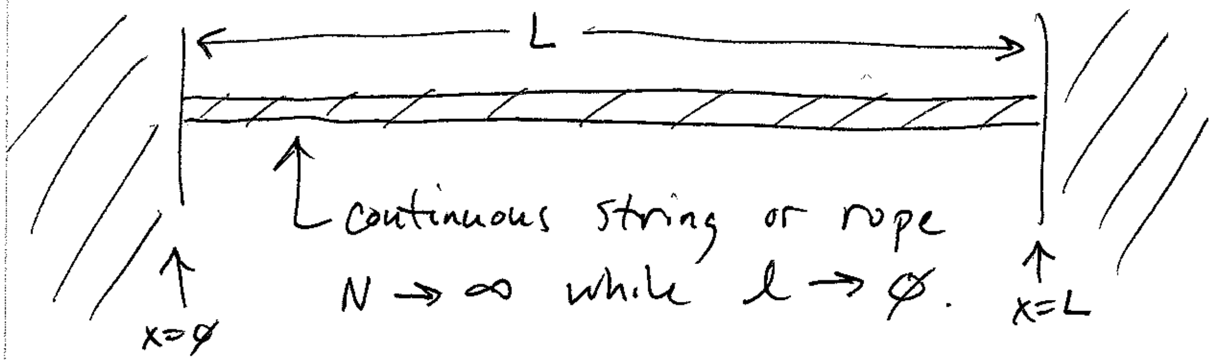
a_n: real part; b_n: imag. part

Then

$$\boxed{a_n = \frac{\vec{y}_0 \cdot \vec{q}_n}{|\vec{q}_n|^2} \quad \text{and} \quad b_n = -\frac{\vec{v}_0 \cdot \vec{q}_n}{\omega_n |\vec{q}_n|^2}}$$

Initial Conditions give the expansion coefficients.

Now we can describe a continuous string by taking the limit where the number of masses goes to infinity while the distance between them goes to zero:



What is the equation of motion for this system?
For N discrete masses we have

$$\ddot{y}_p + 2\omega_0^2 y_p - \omega_0^2 (y_{p+1} + y_{p-1}) = \phi$$

or $$\ddot{y}_p = \omega_0^2 (y_{p+1} - y_p) - \omega_0^2 (y_p - y_{p-1})$$

Recall that $\omega_0^2 = \dfrac{T}{ml}$

So that

$$m \ddot{y}_p = \frac{T}{l}(y_{p+1} - y_p) - \frac{T}{l}(y_p - y_{p-1})$$

Divide by l:

$$\frac{m}{l}\ddot{y}_p = \frac{T}{l}\left(\frac{y_{p+1} - y_p}{l}\right) - \frac{T}{l}\left(\frac{y_p - y_{p-1}}{l}\right)$$

As ℓ goes to zero:

$$\lim_{\ell \to \emptyset} \left(\frac{y_{p+1} - y_p}{\ell} \right) \Rightarrow \lim_{\ell \to \emptyset} \left(\frac{y(x+\ell) - y(x)}{\ell} \right) = \left. \frac{dy}{dx} \right|_{x + \frac{\ell}{2}}$$

Similarly

$$\lim_{\ell \to \emptyset} \left(\frac{y_p - y_{p-1}}{\ell} \right) \Rightarrow \lim_{\ell \to \emptyset} \left(\frac{y(x) - y(x-\ell)}{\ell} \right) = \left. \frac{dy}{dx} \right|_{x - \frac{\ell}{2}}$$

Also let $\frac{m}{\ell} = \rho$ = mass density per unit length

Then

$$\rho \ddot{y}(x) = \frac{T}{\ell} \left[\left. \frac{dy}{dx} \right|_{x+\frac{\ell}{2}} - \left. \frac{dy}{dx} \right|_{x - \frac{\ell}{2}} \right]$$

Take the limit again as $\ell \to \emptyset$

$$\rho \underbrace{\ddot{y}(x)}_{\frac{\partial^2 y}{\partial t^2}} = T \underbrace{\lim_{\ell \to \emptyset} \left[\frac{\left. \frac{dy}{dx} \right|_{x+\frac{\ell}{2}} - \left. \frac{dy}{dx} \right|_{x-\frac{\ell}{2}}}{\ell} \right]}_{\frac{\partial^2 y}{\partial x^2}}$$

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T} \frac{\partial^2 y}{\partial t^2}}$$ Classical Wave Equation.

The classical wave equation is the equation of motion for the continuous ~~str~~ string. It's just Newton's 2^{nd} Law for a continuous system.

Solution for the string fixed at $x=\emptyset$ and $x=L$.

We'll use the solution of the loaded string and take the limit where $N \to \infty$, $\ell \to \emptyset$ such that $(N+1)\ell = L$ = total length.

For N particles,

$$y_{pn} = C_n \sin\left(\frac{pn\pi}{N+1}\right)$$

* Now let $p\ell = x$ = distance along the string. Then

$$y_n(x) = C_n \sin\left(\frac{\overbrace{(p\ell)}^{x}\pi n}{\underbrace{(\ell)(N+1)}_{L = \text{total length}}}\right) = \boxed{C_n \sin\left(\frac{n\pi x}{L}\right)}$$

$n = 1, 2, 3, \ldots, \infty.$

Normal Modes for the string fixed at $x=\emptyset$ and $x=L$.

The normal mode frequencies are

$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right) \quad \leftarrow \text{loaded string frequencies}$$

$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi l}{2\underbrace{(N+1)l}_{L}}\right) = 2\omega_0 \sin\left(\frac{n\pi l}{2L}\right)$$

Now consider the limit where $l \to \emptyset$.
Also let n be small but non-zero. Then

$$\omega_n = 2\omega_0\left[\lim_{l\to\emptyset} \sin\left(\frac{n\pi l}{2L}\right)\right] \approx 2\omega_0\left(\frac{n\pi l}{2L}\right)$$

since $\sin\left(\frac{n\pi l}{2L}\right) \approx \frac{n\pi l}{2L}$

for small l.

Then $$\omega_n = \frac{\omega_0 n\pi l}{L}$$

* Simplify: $$\omega_0 = \sqrt{\frac{T}{ml}} = \sqrt{\frac{T/l^2}{m/l}} = \frac{1}{l}\sqrt{\frac{T}{\rho}}$$

where $\rho = \frac{m}{l}$

Then $$\boxed{\omega_n = \sqrt{\frac{T}{\rho}}\ \frac{n\pi}{L}}\ , \quad n = 1, 2, 3, \ldots\ldots$$

↑ Normal mode eigenfrequencies for the string fixed at $x=\emptyset$ and $x=L$.

The normal modes are

$$y_n(x,t) = c_n \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n t},$$

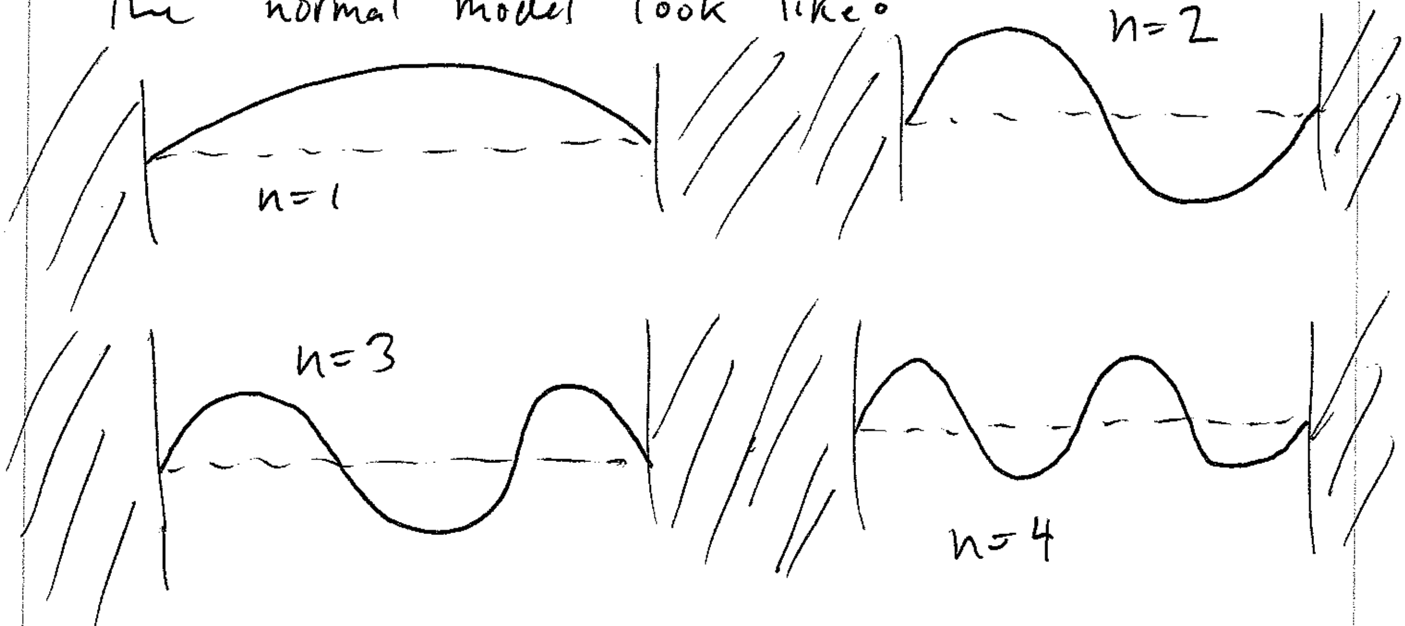
where $\omega_n = \sqrt{\frac{T}{\rho}}\ \frac{n\pi}{L}$

The general solution is a sum over normal modes:

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} c_n \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{\text{normal mode}} \underbrace{e^{i\omega_n t}}_{\text{time evolution}}}$$

c_n: expansion coefficients, determined by initial conditions.

The normal modes look like:



Optional Demo: G3-21 - Transverse Waves on a long spring. Illustrate the first few normal modes.

Calculating the expansion coefficients from the initial conditions

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The $\{c_n\}$ have real and imaginary parts:

$$c_n \equiv a_n + i b_n.$$

The $\{a_n\}$ are determined by the initial position of the string at $t=0$:

$$y(x,t=0) = \mathrm{Re}\left[\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n (0)}\right]$$

$t=0$

$$= \sum_{n=1}^{\infty} \mathrm{Re}(c_n) \sin\left(\frac{n\pi x}{L}\right)$$

$$\boxed{y(x,t=0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)}$$

Similarly the $\{b_n\}$ are related to the initial velocity of the string:

$$\dot{y}(x,t=0) = \mathrm{Re}\left[\sum_{n=1}^{\infty} c_n (i\omega_n) \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n (0)}\right]$$

$t=0$

$$\dot{y}(x, t=\emptyset) = \sum_{n=1}^{\infty} \omega_n \underbrace{\mathrm{Re}(ic_n)}_{-b_n} \sin\left(\frac{n\pi x}{L}\right)$$

$$\boxed{\dot{y}(x, t=\emptyset) = \sum_{n=1}^{\infty} -\omega_n b_n \sin\left(\frac{n\pi x}{L}\right)}$$

Our job is to determine the $\{a_n\}$ and $\{b_n\}$ given the initial conditions $y(x, t=\emptyset)$ and $\dot{y}(x, t=\emptyset)$.

In fact, its easy to calculate the expansion coefficients because the normal modes are orthogonal:

$$\boxed{\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} \emptyset, & m \neq n \\ \frac{L}{2}, & m = n \end{cases}}$$

Or we can write this using the Kronecker Delta:

$$\boxed{\frac{2}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \delta_{mn}}$$

where $\delta_{nm} = \begin{cases} \emptyset, & n \neq m \\ 1, & m = n \end{cases}$

Suppose we want to know what the ~~first~~ seventh $\{a_n\}$ value is. (a_7). We can calculate it like this:

Evaluate $\displaystyle \int_0^L \sin\left(\frac{7\pi x}{L}\right) \underbrace{y(x,t=\emptyset)}\, dx$

$$y(x,t=\emptyset) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

$$= \int_0^L \sin\left(\frac{7\pi x}{L}\right) \left[\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)\right] dx$$

$$= \sum_{n=1}^{\infty} a_n \underbrace{\int_0^L \sin\left(\frac{7\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx}_{\frac{L}{2}\delta_{n7}}$$

$$= \sum_{n=1}^{\infty} a_n \left(\frac{L}{2}\delta_{n7}\right)$$

Kronecker Delta kills all terms in the sum except the $n=7$ term:

$$= a_7\left(\frac{L}{2}\right)$$

$$\therefore \boxed{a_7 = \frac{2}{L}\int_0^L \sin\left(\frac{7\pi x}{L}\right) y(x,t=\emptyset)\, dx}$$

In general, to calculate coefficient a_n, we should evaluate this integral:

$$\boxed{a_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) y(x, t=0)\, dx}$$

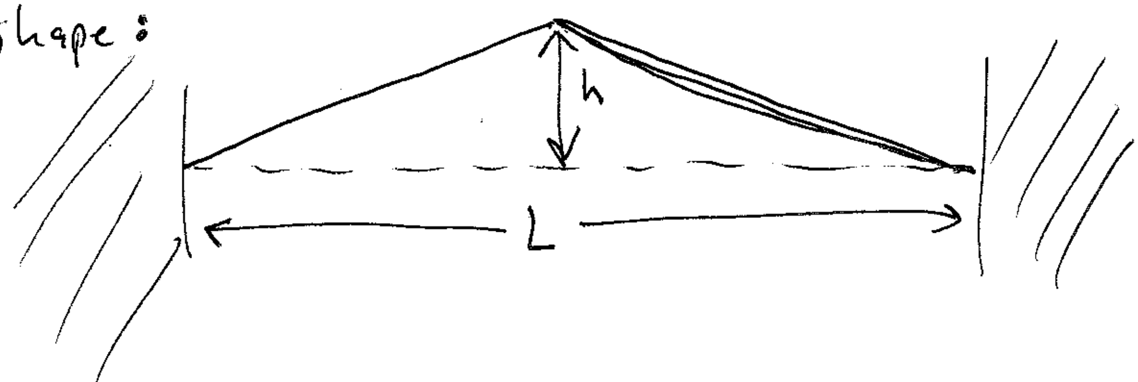
Note that $n = 1, 2, 3, \ldots \infty$, so in general we may need to evaluate an infinite number of integrals (one integral for each coefficient).

The $\{b_n\}$ can be calculated in a similar way from the initial velocity:

$$\boxed{b_n = \frac{-2}{\omega_n L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) \dot{y}(x, t=0)\, dx}$$

Example: Triangular String (Plucked String).

Suppose at $t=0$ the string has a triangular shape:



And we release it from rest and allow it to evolve in time according to the equation of motion. What is the time dependent solution?

Solution

First, since we release it from rest,

$$\dot{y}(x, t=\emptyset) = \emptyset$$

Then the $\{b_n\}$ must be zero:

$$b_n = -\frac{2}{\omega_n L}\int_{\emptyset}^{L} \sin\left(\frac{n\pi x}{L}\right) \underbrace{\dot{y}(x, t=\emptyset)}_{\emptyset} \, dx \; \boxed{= \emptyset}$$

for all n.

The a_n can be calculated according to

$$a_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) y(x, t=\emptyset)\, dx$$

The initial position function is

$$y(x, t=\emptyset) = \begin{cases} \left(\frac{2h}{L}\right) x, & \emptyset \le x \le \frac{L}{2} \\ \left(\frac{2h}{L}\right)(L-x), & \frac{L}{2} \le x \le L \end{cases}$$

so

$$a_n = \frac{2}{L}\int_0^{\frac{L}{2}} \sin\left(\frac{n\pi x}{L}\right)\left(\frac{2hx}{L}\right) dx + \frac{2}{L}\int_{\frac{L}{2}}^{L} \sin\left(\frac{n\pi x}{L}\right)\left(\frac{2h(L-x)}{L}\right) dx$$

This is a simple calculus problem.

Let $\quad x' \equiv x - \frac{L}{2}$

so that $\quad x = x' + \frac{L}{2}$

The initial position function can be written in terms of x':

$$y(x', t=\emptyset) = \begin{cases} \frac{2h}{L}\left(x' + \frac{L}{2}\right) & , \quad -\frac{L}{2} \le x' \le \emptyset \\ \frac{2h}{L}\left(-x' + \frac{L}{2}\right) & , \quad \emptyset \le x' \le \frac{L}{2} \end{cases}$$

Note that $y(x', t=\emptyset)$ is an <u>even</u> function of x'.

Also, we have the following trig identity:

If $x = x' + \frac{L}{2}$, then

$$\sin\left(\frac{n\pi x}{L}\right) = \begin{cases} (-1)^{(n-1)/2} \cos\left(\frac{n\pi x'}{L}\right) & , \quad \text{for } n=\text{odd} \\ (-1)^{n/2} \sin\left(\frac{n\pi x'}{L}\right) & , \quad \text{for } n=\text{even} \end{cases}$$

Now our integral has 2 cases:

For $n=$ odd:

$$a_n = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} y(x') (-1)^{(n-1)/2} \cos\left(\frac{n\pi x'}{L}\right) dx'$$

and for $n=$ even:

$$a_n = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \underbrace{y(x')}_{\substack{\text{even} \\ \text{function} \\ \text{of } x'}} (-1)^{n/2} \underbrace{\sin\left(\frac{n\pi x'}{L}\right)}_{\substack{\text{odd function} \\ \text{of } x'}} dx' \; \boxed{= \emptyset}$$

because the integrand is odd and the interval $-\frac{L}{2}$ to $\frac{L}{2}$ is symmetric.

So we only need to evaluate the $n = \text{odd}$ case:

$$a_n = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \underbrace{y(x')\,(-1)^{(n-1)/2}\cos\left(\frac{n\pi x'}{L}\right)}\,dx', \quad n = \text{odd}$$

this integrand is an
<u>even</u> function of x'
so we can evaluate it
from zero to $\frac{L}{2}$ and
multiply by 2.

$$a_n = (2)\left(\frac{2}{L}\right)\int_0^{\frac{L}{2}} y(x')\,(-1)^{(n-1)/2}\cos\left(\frac{n\pi x'}{L}\right)dx'$$

integration by parts.

$$= \frac{4}{L}(-1)^{(n-1)/2}\left(\frac{2h}{L}\right)\int_0^{\frac{L}{2}}\left(-x' + \frac{L}{2}\right)\cos\left(\frac{n\pi x'}{L}\right)dx'$$

$$= \frac{8h}{L^2}(-1)^{(n-1)/2}\left[-\left(\frac{L}{n\pi}\right)^2\cos\left(\frac{n\pi x'}{L}\right) - \frac{x'L}{n\pi}\sin\left(\frac{n\pi x'}{L}\right) + \left(\frac{L}{2}\right)\left(\frac{L}{n\pi}\right)\sin\left(\frac{n\pi x'}{L}\right)\right]\Bigg|_{\varnothing}^{\frac{L}{2}}$$

zero for $n = \text{odd}$ — cancel

$$= \frac{8h}{L^2}(-1)^{(n-1)/2}\left[-\left(\frac{L}{n\pi}\right)^2\underbrace{\cos\left(\frac{n\pi}{2}\right)}_{\text{zero for } n=\text{odd}} - \frac{L^2}{2n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{L^2}{2n\pi}\sin\left(\frac{n\pi}{2}\right) - \left(-\left(\frac{L}{n\pi}\right)^2\right)\right]$$

$$\boxed{= \frac{8h}{(n\pi)^2}(-1)^{(n-1)/2}, \quad n = \text{odd}}$$

In other words,

$$a_1 = \frac{8h}{\pi^2}, \quad a_2 = \emptyset, \quad a_3 = \frac{-8h}{9\pi^2}, \quad a_4 = \emptyset,$$

$$a_5 = \frac{8h}{25\pi^2}, \quad a_6 = \emptyset, \quad \ldots\ldots$$

So the final solution for these initial conditions is

$$\boxed{y(x,t) = \sum_{\substack{n=1 \\ \text{odd } n \text{ only.}}}^{\infty} \left(\frac{8h}{(n\pi)^2} (-1)^{(n-1)/2} \right) \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n t}}$$

where $\omega_n = \sqrt{\frac{T}{\rho}}\, \frac{n\pi}{L}$, $n = 1, 3, 5, 7, \ldots$