Physics 273 - Homework #5

1) Consider two coupled mechanical oscillators. Each of the two masses are connected to a fixed wall through a spring with spring constant (k). The masses are connected to each other with a spring with spring constant (k_{12}).

a) Suppose we hold one of the two masses fixed, so it cannot move. What is the natural frequency of the other mass?

b) Now we release the second oscillator. Compare the natural frequency from part (a) to the two normal mode frequencies of the double oscillator. Are they smaller or larger? Explain.

c) What happens to the two normal mode frequencies of the system in the limit where the coupling spring constant (k_{12}) becomes very large and very small? Explain.

2) Consider again two coupled mechanical oscillators. As usual, each oscillator is connected to a wall by a spring with spring constant (k), and the two masses are coupled together by a spring with spring constant (k_{12}). Suppose that our initial conditions are that the first oscillator has an initial velocity of (v_0) and an initial position of zero, and the second oscillator starts with an initial velocity and position of zero.

a) Apply these initial conditions to find $x_1(t)$ and $x_2(t)$.

b) Let k = 1 N/m, k_{12} = 0.25 N/m, m = 1 kg, and v_0 = 1 m/s. Make a plot of the positions of both masses from t = 0 to t = 30 seconds. Please put both $x_1(t)$ and $x_2(t)$ on the same plot.

3) For the loaded string, the amplitude relationships which define the normal modes are described by the expression:

$$A_{pn} = \sin\left(\frac{pn\pi}{N+1}\right)$$

where (p) tells us which mass we are talking about, (n) tells us which normal mode we are talking about, and N is the total number of masses on the string. It is convenient to re-write this expression in a vector notation:

$$\vec{q}_n = \left(\sin\left(\frac{n\pi}{N+1}\right), \sin\left(\frac{2n\pi}{N+1}\right), \sin\left(\frac{3n\pi}{N+1}\right), \ldots, \sin\left(\frac{Nn\pi}{N+1}\right)\right)$$

In this expression, the first component of the vector describes mass #1, the second component describes mass #2, ect. Since there are N normal modes, there will be N such vectors. These are the normal mode eigenvectors.

a) Write down in explicit numerical form the vectors $\vec{q}_1$ and $\vec{q}_2$ for the N = 2 case. Please give a numerical value for each component of the vectors, accurate to three decimal places.

b) Calculate the dot products for all pairs of vectors for the N = 2 case: $\vec{q}_1 \cdot \vec{q}_1$, $\vec{q}_1 \cdot \vec{q}_2$, and $\vec{q}_2 \cdot \vec{q}_2$.

c) Repeat part (a) for the N = 3 case, writing down the explicit numerical form for the three eigenvectors.

d) Repeat part (b) for the N = 3 case. You will need to calculate six unique dot products.

e) Repeat part (a) for the N = 4 case, writing down the explicit numerical form for the four eigenvectors.

f) Repeat part (b) for the N = 4 case. You will need to calculate the 10 unique dot products.

g) What pattern do you see in the dot products of these eigenvectors?

4) Consider a loaded string consisting of three particles of mass (m) regularly spaced on the string. At t = 0 the center particle is displaced a distance (a) from its equilibrium position. (The other two particles are located at their equilibrium positions.) We release all three particles with an initial velocity of zero.

a) Apply these initial conditions to the solution of the loaded string (which we found in class) to find the position of all three particles as a function of time.

b) Let the string tension be T = 10 N, m = 1 kg, and let the distance between the masses on the string be 0.1 m. Also let the initial displacement of mass 2 be 0.01 m. Make a plot of the positions of all three masses from t = 0 to t = 10 seconds. Please put $x_1(t)$, $x_2(t)$, and $x_3(t)$ on the same plot.

5) **Ringing and damping of a mechanical oscillator (numerical).** Make a copy of your numerical solution to Homework #4 problem #2 (forced oscillator with damping). In this problem we will change the forcing function to the following step function (or square wave function):

$$F(t) = \begin{cases} 1.0 \text{ Newtons}, & 0 < t < 10 \text{ s} \\ 0.0 \text{ Newtons}, & 10 \text{ s} < t < 20 \text{ s} \end{cases}$$

This forcing function repeats itself thereafter with a period of 20 seconds. Hint: If you are using excel, one way to implement this forcing function in your numerical calculation is to create a new column which contains the value of the forcing function at each moment in time. You can then reference this column when calculating the acceleration.

a) Let x0 = 0.0 m, v0 = 0.0 m/s, m = 1 kg, k = 30 N/m, and b = 2 N/(m/s). Use a time step of 0.01 seconds, and calculate for 4000 steps (a total of 40 seconds). Print out a plot the position of the oscillator as a function of time.

b) In your solution you should see a phenomena called "ringing". Measure the angular frequency of the ringing, and compare it to the oscillation frequency that you would expect for this oscillator.

c) The frequency of a damped oscillator is given by

$$\omega_d = \sqrt{\omega_0^2 - \gamma^2/4}$$

Suppose we increase the drag coefficient (b or γ) until $\omega_d = 0.0$. This condition is called "critical damping", and it is the condition for all oscillations to be eliminated. Calculate the value of the drag coefficient (b) which achieves critical damping for the oscillator parameters described in part (a).

d) Starting with a drag coefficient of b = 2 N/(m/s), increase (b) one unit at a time up to 20 N/(m/s), and observe the effect on the oscillator's position as a function of time. Now set (b) equal to the critical value that you calculated in part (c), and print out a plot of the oscillator's position for this situation.

Comment: In real mechanical and electrical systems where oscillations are undesirable, the components will often be chosen so that the system is critically damped. For example, a bridge might be designed to be critically damped to prevent large oscillations in the event of an earthquake.

e) When the drag coefficient is increased beyond the critical value, we say that the system is "over-damped". Just like critically damped systems, over-damped systems also do not oscillate, however, they take longer to return to equilibrium after a shock. To see the response of an over-damped system, set b = 50 N/(m/s), and print out a plot of the oscillator's position as a function of time.